Equidistribution in Number Theory

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Lecture 1 6/9/2018

These are notes for Ali Altuğ's course MA841 at BU Fall 2018.

The course webpage is http://math.bu.edu/people/saaltug/2018_3/2018_3_sem.html.

Course overview: Upon Akshay Venkatesh winning the fields medal I decided to dedicate this term to various aspects of equidistribution results in number theory and their relations to L-functions. I am aiming to cover basic results like Linnik's and Duke's theorems, as well as certain aspects of subconvexity. As we move along we may briefly touch several other aspects like quantum unique ergodicity (QUE) or equidistribution in other settings (e.g. over function fields). There is an abundance of material on this topic. We will not follow a single book or an article (although the first book and the survey articles following that will be the main reference), however here are a bunch of helpful papers/books. I will update these references as we move along.

1 Equidistribution in Number Theory

1.1 Introduction

> I wasn't going to do this at all, but then Akshay won the Fields medal.
>
> —Ali Altuğ

Some topics we will try and cover:

1. Linnik's and Duke's theorem 1970-1990
2. lattice points and hyperbolic geometry
3. modular forms
4. spectral theory
5. modular forms of half-integral weight.
6. harmonic analysis
7. Kuznetsov formula
8. Other topics!

1.2 Linnik and Duke

Akshay's first breakthrough, according to Sarnak, was subconvexity. He became interested in ergodic theory, because they could prove hard theorems, equidistribution is a powerful tool in number theory.

Our main goal will be to talk about: Some problems stated (and proved) by Linnik. In the book, Ergodic properties of algebraic fields, 1968. He considered lattice points on a sphere of radius n, these are points (say in $\mathbf{R}^3$) whose coordinates are integral of a fixed distance from the origin. Analogous to the circle problem. If n is fixed there is nothing to distribute, but if we vary n and project down we can ask do they accumulate miss any patches, generally how do they distribute. Of course this can be generalised.

Setup. Sphere S^2.

Lattice points $\alpha = (x_1, x_2, x_3) \in \mathbf{Z}^3$.

$$|\alpha|^2 = x_1^2 + x_2^2 + x_3^2$$

this is where the number theory comes, we are looking at representability of numbers n by this ternary quadratic form.

Various methods exist for studying this for varying n, quadratic reciprocity for $n = 2$, circle method/Vinogradov for $n = 4$. $n = 3$ is the cut off, here half-integral weight modular forms are relevant.

Set

$$\Omega_N = \left\{ x = \frac{\alpha}{|\alpha|} : \alpha \in \mathbf{Z}^2,\ |\alpha|^2 = N \right\} \subseteq S^2.$$

Question 1.1 Are Ω_N "equidistributed" as $N \to \infty$? □

There is an immediate obstruction to N being a sum of 3 squares, e.g. $N = 7$ implies $\Omega_7 = \emptyset$.

Recall: (Gauss/Legendre)

$$N = x_1^2 + x_2^2 + x_3^2,\ x_i \in \mathbf{Z} \iff N \neq 4^a(8b + 7),$$

so we avoid these sets and ask the same question.

Theorem 1.2 Linnik. *Let $f \in C^\infty(S^2)$, then as $N \to \infty$, N squarefree $N \not\equiv 7 \pmod 8$, $\left(\frac{N}{p}\right) = 1$ for some fixed odd prime p*

$$\frac{1}{\#\Omega_N} \sum_{x \in \Omega_N} f(x) \to \int_{S^2} f \, \mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the Lebesgue measure on S^2.

This is saying that the points are equidistributed with respect to the Lebesgue measure.

The last condition is a defect of the method, known as a Linnik condition.

Remark 1.3 Linnik's proof is ergodic theoretic.

After this came Duke in 1988, in the mean time, Weil conjectures were proved, Iwaniec gave bounds for Kloosterman sums. Duke was a graduate student of Sarnak at Courant. He gave a more direct proof of 1.2 which does not have the $\left(\frac{N}{p}\right) = 1$ condition. His proof is based on the theory of

(half-integral weight) modular forms, and a good bound Iwaniec on certain exponential sums.

Why do exponential sums enter the picture? We are trying to prove that we have a sum converging to an integral. Generally we work with a basis of functions first, we could try using fourier analysis, using harmonics as our basis, this is when exponential sums appear. That requires us to work out harmonics on the sphere (spherical harmonics) which leads to representation theory, which as the sphere is compact involves Weyl representations etc.

Duke also proved "the same" theorem over modular surfaces. Instead of looking at expanding spheres we study expanding hyperboloids.

Setup. Space

$$\pm\Gamma\backslash\mathbf{H}$$

$$\mathbf{H} = \left\{x + iy \in CC : y > 0\right\}$$

$$\Gamma = \mathrm{SL}_2(\mathbf{Z})$$

$$\mathcal{F} = \left\{z \in \mathbf{H} : -\frac{1}{2} \leq \Re(z) \leq 0, |z| \geq 1 \text{ or } \frac{1}{2} > \Re(z) > 0, |z| > 1\right\}$$

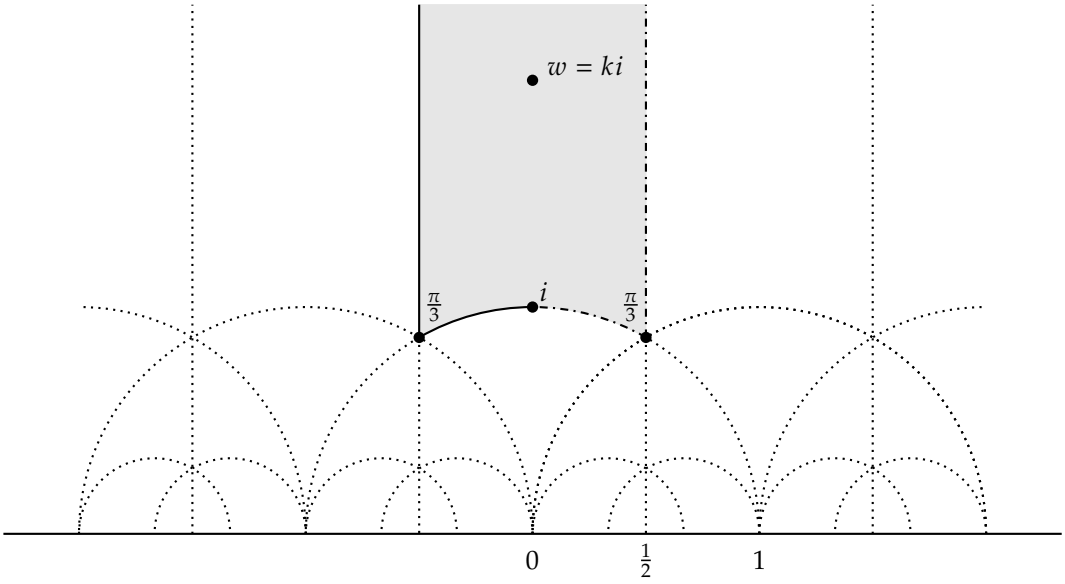

Figure 1.4: $\mathcal{F}$

Ω_N's are now replaced with CM points.

Digression (CM points). Q a binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2$$

of discriminant $d = b^2 - 4ac < 0$, $a, b, c \in \mathbf{Z}$.

Theorem 1.5 Gauss. *There are only finitely many equivalence classes of such forms for fixed d.*

The number of such is given by the Hurwitz class number.
Given Q we associate a CM point

$$z_Q = \frac{-b + \sqrt{d}}{2a} \in \mathbf{H}.$$

The action of Γ on Q is the same as the action of Γ on z_Q.

$$z_{\gamma Q} = \gamma z_Q.$$

Let

$$\Lambda_d = \{z_Q \in \mathcal{F} : \mathrm{disc}(Q) = d\}.$$

We need to sum properly to take automorphisms into account.

Definition 1.6

$$\sum_{z_Q \in \Lambda_d}^{*}$$

is the sum weighted by $\frac{1}{2}$ if $Q = a(x^2 + y^2)$ $(d = -4)$, $\frac{1}{3}$ if $Q = a(x^2 + xy + y^2)$ $(d = -3)$. ◊

If we want to be fancy we can say the word stack here.

Remark 1.7

$$\sum_{z_Q \in \Lambda_d}^{*} 1 = H(d)$$

where $H(d)$ is the Hurwitz class number.

Remark 1.8 If d is fundamental, i.e. a discriminant of some $\mathbf{Q}(\sqrt{d})$ then $H(d) = h(d)$ the regular class number.

For the measure on $\mathcal{F}$ we take

$$\mathrm{d}\mu = \frac{3}{\pi} \frac{\mathrm{d}x\, \mathrm{d}y}{y^2}.$$

Theorem 1.9 Duke. *Let $f \in C^\infty(\mathbf{H})$, that is Γ invariant and bounded on $\mathcal{F}$, then as $d \to \infty$ over fundamental discriminants*

$$\frac{1}{\#\Lambda_d} \sum_{z \in \Lambda_d}^{*} f(z) \to \iint_{\mathcal{F}} f \, \mathrm{d}\mu.$$

Equidistribution implies density, but is so much more, for example we cannot have dense points but which happen to cluster towards some line for example.

1.3 Basics

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Question: Let $\alpha \in \mathbf{R}$ and consider $\{\alpha n\}$ where $\{x\} = x \pmod 1$ so $\left\{\frac{3}{2}\right\} = \frac{1}{2}$. How are these distributed?

Example 1.10 If $\alpha = \frac{2}{7}$ then we have $\{\{\alpha n\} : n \in \mathbf{N}\} = \{\frac{i}{7} : i \in \{0, \ldots, 6\}\}$ and in fact it hits each evenly. □

Example 1.11 If $\alpha = \sqrt{2}$ so $\{\alpha\} \approx 0.4142?...$ $\{\alpha 2\} \approx 0.8284?...$ $\{\alpha 3\} \approx 0.24264...$ $\{\alpha 4\} \approx 0.656854...$ These spread out densely, but there is a difference between density and equidistribution. In this example, equidistribution says that the

proportion of time the sequence spends in each interval (a, b) is $b - a$. □

So questions are: is $\{n\alpha\}$ dense?

Is $\{n\alpha\}$ uniformly distributed (equidistributed with respect to the standard measure)?

The answer to both questions is yes.

Theorem 1.12 Kronecker. *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ then $\{n\alpha\}$ is dense in $[0, 1)$.*

Digression (Diophantine approximation). This is a very tough area of number theory, not so many definitive results here.

Theorem 1.13 Dirichlet. *Let $\alpha \in \mathbf{R}, N \in \mathbf{Z}_{>0}$ then there exists p, q with $q > 0$.*

$$\left|q\alpha - p\right| < \frac{1}{N}.$$

Proof. (Pidgeonhole) divide $[0, 1)$ into even N subintervals of width $\frac{1}{N}$, consider

$$\alpha_0 = 0, \alpha_1 = \{\alpha 1\}, \alpha_2 = \{\alpha 2\}, \ldots \in [0, 1)$$

as soon as we get to α_N we must have two in one subinterval say $|\alpha_{n_1} - \alpha_{n_2}| < \frac{1}{N}$. So there exists p_{n_1}, p_{n_2} such that

$$|n_1\alpha - k_{n_1} - (n_2\alpha - k_{n_2})| < \frac{1}{N}$$

$$|(n_1 - n_2)\alpha - (k_{n_1} - k_{n_2})| < \frac{1}{N}.$$

■

Corollary 1.14 *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ then there exists infinitely many coprime p, q with $q > 0$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Proof. Exercise. ■

This is very strong, Roth's theorem tells us that even $q^{2+\epsilon}$ here is enough to force finiteness.

Note 1.15 $\alpha \notin \mathbf{Q}$ is necessary! Otherwise

$$\left|\frac{p_0}{q_0} - \frac{p}{q}\right| = \left|\frac{p_0 q - p q_0}{q q_0}\right| \geq \frac{\left|p_0 q - p q_0\right|}{\max\left\{q^2, q_0^2\right\}}$$

so choose $q > q_0$ implies $\frac{p_0}{q_0} = \frac{p}{q}$.

One can do better:

Theorem 1.16 Hurwitz. *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

then there exists infinitely many coprime p, q with $q > 0$ satisfying.

Note 1.17 $\sqrt{5}$ is the best possible without further restriction on α.

Example 1.18 If $\alpha = \frac{1-\sqrt{5}}{2}$ then

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Aq^2}$$

has only finitely many solutions for $A > \sqrt{5}$. □

What if we allow further restriction?

Theorem 1.19 Liouville. *Let $\alpha \in \mathbf{R}$ algebraic of degree $n > 1$. Then there exists $A > 0$ such that for all p, q with $g > 0$*

$$\left|\alpha - \frac{p}{q}\right| > \frac{A}{q^n}.$$

Proof. Let $f \in \mathbf{Z}[x]$ be the minimal polynomial of α. Gauss's lemma implies that f is irreducible over $\mathbf{Q}$ so

$$q^n f\left(\frac{p}{q}\right) \in \mathbf{Z} \setminus \{0\} \; \forall \frac{p}{q} \in \mathbf{Q}.$$

The mean value theorem says that there exists $x_0 \in [\alpha, p/q)$ s.t.

$$\frac{f(p/q) - \cancelto{0}{f(\alpha)}}{p/q - \alpha} = f'(x_0)$$

so

$$\frac{q^n f(p/q)}{q^n f'(x_0)} = |p/q - \alpha|,$$

notice how n appears here. ■

This theme of using some calculus is repeated across diophantine analysis.

Remark 1.20

- Thue: can replace n with $(\deg(\alpha) + 2)/2$ (This already has implications to integral solutions of degree ≥ 3 polynomials $f \in \mathbf{Z}[x]$, e.g. elliptic curves with bounded integral discriminant)
- Roth ($\sim$ 1958): for all $\epsilon > 0$, there are only finitely many p/q satisfying

$$|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}}.$$

Back to equidistribution. $a_n = \{n\alpha\}, \{a_n\}_{n=1}^{\infty}$.

Theorem 1.21 Kronecker. *a_n is dense in $[0, 1)$ if $\alpha \notin \mathbf{Q}$.*

Proof. Will show for any $x \in [0, 1)$ there exists

$$\left\{a_{n_j}\right\}_{j=1}^{\infty} \text{ s.t. } a_{n_j} \to x.$$

Notation $\|x\|$ means the distance to the nearest integer. Dirichlet implies that infinitely many p/q have $\alpha - p/q| < 1/q^2$.Given $\epsilon > 0$ let q be such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2} \implies |q\alpha - p| < \frac{1}{q}$$

$$\frac{1}{q} < \epsilon$$

Choose j such that $j(\alpha q - p)$ is within $1/q$ of x (why?). So

$$\|j(\alpha q - p) - x\| < \frac{1}{q} < \epsilon.$$

(Fill in the gaps here). ■

Now let's define what is means to be uniformly distributed.

Definition 1.22 Uniformly distributed sequence. $a_1, a_2, \ldots$ is called **uniformly distributed** if for all $(b, c) \subseteq [0, 1)$

$$\#\{n \leq N : \{a_n\} \in (b, c)\} \sim N(c - b)$$

or

$$\lim_{N\to\infty} \frac{\#\{n \leq N : \{a_n\} \in (b, c)\}}{N} = c - b.$$

◊

Uniformly distributed or not?

When $\alpha \in \mathbf{R} \setminus \mathbf{Q}$: $\{n\alpha\}$, $\{n^2\alpha\}$, $\{(3n^2+2n+1)\alpha\}$ $\{(\sqrt{5}n^3+2n-(\zeta_{10}+\bar{\zeta}_{10}))\alpha\}$ all are.

$\{n!e\}$, $\{\log(n)\}$ and $\{\log(p_n)\}$ are not.

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Example 1.23 Several.

1. $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, $a_n = \{n\alpha\}$. Yes
2. $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, $a_n = \{n^2\alpha\}$. Yes
3. $a_n = \{\log(n)\}$. No
4. $a_n = \{n!e\}$. No (not dense).
5. $\{\log(p_n)\}$. No.
6. $a_n = \{\sqrt{n}\}$. Yes.
7. $a_n = \{e^n\}$. ?
8. $a_n = \{\log n!\}$. Yes ?
9. $a_n = \{\log\log n!\}$. No

To begin let's show $a_n = \{n!e\}$ is not equidistributed as it has only one limit point 0.

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}$$

so $n!e \in \mathbf{Z} + \frac{1}{n+1} + \frac{(n+1)(n+2)}{+} \cdots \leq \frac{e}{n+1} \to 0$, as $n \to \infty$. So its not dense, hence certainly not equidistributed. □

Theorem 1.24 *The sequence $a_n = \{n\alpha\}$ is equidistributed.*

Proof. Let $\epsilon > 0$, choose $M > 0$ such that $\frac{1}{M} < \epsilon$, furthermore choose a $\delta < \frac{1}{M}$. Dirichlet implies there exists $m \in \mathbf{Z}$ s.t. $\|m\alpha\| = \delta$.

$$0 < \delta < \frac{1}{M} < \epsilon$$

recall $\|x\|$ is the distance to the nearest integer. Consider the set $S_i = \{\{n\alpha\} : n < N,\ n \equiv i \pmod m\}$ then

$$S = \{\{n\alpha\} : n < N\} = \bigsqcup_{i=1}^{m} S_i$$

moreover,

$$S_i = \{\{km\alpha + i\alpha\} : 0 \leq k \leq N_i\}$$

where

$$N_i = \frac{N}{m} + O(1)$$

(why is the $O(1)$ here?). We can rewrite these S_i as follows:

$$S_i = \{\delta k + \gamma_i \pmod 1 : 0 \leq k \leq N_i\}$$

where

$$\begin{cases} i\alpha \pmod 1 & \text{if } \delta = \{m\alpha\} \\ i\alpha - \delta(N_i + 1) \pmod 1 & \text{if } \delta = 1 - \{m\alpha\} \end{cases}$$

replace k by $N_i + 1 - k$.
Now $0 \leq \gamma < 1$ and let

$$K_i = \lfloor \delta N_i + \gamma_i \rfloor$$

$$\#\{k < N_i : \{\delta k + \gamma_i\} \in [b, c]\} = \sum_{j=0}^{K_i} \#\{k \leq N_i : \delta k + \gamma \in [j + b, j + c]\}$$

$$= (K_i + O(1)) \left(\frac{c - b}{\delta} + O(1) \right)$$

$$N_i(c - b) + O\left(\frac{c - b}{\delta} + \delta N_i + 1 \right)$$

so

$$S = \bigsqcup_{i=1}^{m} S_i$$

$$\#S = N(c - b) + O\left(\frac{M}{\delta} + \delta N \right)$$

(check!). Finally choose $N > \frac{M}{\delta^2}$. So that

$$\frac{M}{\delta} + \delta N < \epsilon N.$$

Which implies

$$\#\{n < N : \{n\alpha\} \in [b, c]\} = \#S = N(c - b) + \epsilon N.$$

■

We followed our nose essentially, but we needed to put ourselves in a favourable position so that we can get a good handle on the error.

Example 1.25 10. Consider the sequence

$$b_{m,i} = \frac{i}{m}$$

and define

$$a_{\binom{m}{2}+i} = b_{m,i}$$

so that

$$b_{m,i}$$

$$0, 1$$

$$0, \frac{1}{2}, 1$$

$$0, \frac{1}{3}, \frac{2}{3}, 1$$

$$0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$$

$$a_m = 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$$

this is uniformly distributed! □

Exercise 1.26 Prove this (in a similar way to before).

Proposition 1.27

1. *Any sequence has a rearrangement that is not uniformly distributed.*
2. *Any dense sequence has a rearrangement that is uniformly distributed.*

Proof.

1. If not sense we are done already. Otherwise let $I_1 = [0, \frac{1}{2}), I_2 = [\frac{1}{2}, 1]$ Let $X = \{a_n \in I_1\}, Y = \{a_n \in I_2\}$, so both sets are infinite. Let $b_n = x_1, \ldots, x_{10^{10}}, y_1, x_{10^{10}+1}, \ldots, x_{20^{20}}, \ldots$ so $\#\{n : b_n \in [4/5, 1]\} \sim \frac{N}{10^{10}}$.
2. If a_n is dense then, let

$$b_{n,i} = a_m \text{ s.t. } \frac{i-1}{n} < a_m < \frac{i}{n}$$

by the example above this is equidistributed.

■

What can we say about the space of uniformly distributed sequences? Is it closed under addition? No ($a_n = -b_n$). $\{ka_n\}$ neither ($a_n = \alpha n$, $\frac{1}{\alpha}a_n = n$). $a_n b_n$ doesn't work either, but if b_n converges then $a_n b_n$ is uniformly distributed.

1.4 Weyl's criterion (1916)

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Theorem 1.28 Weyl's criterion. $\{a_n\}_{n=1}^{\infty} \in \mathbf{R}$ *then TFAE:*

1. $\{a_n\}_{n=1}^{\infty}$ *is uniformly distributed mod* 1.
2. $\forall f \in C([0,1])$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{a_n\}) = \int_0^1 f \, \mathrm{d}x.$$

3.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(ma_n) = \delta_m,\ \forall m \in \mathbf{Z}$$

where $e(x) = e^{2\pi i x}$,

$$\delta_m = \begin{cases} 0 & m \neq 0 \\ 1 & m = 0 \end{cases},$$

collectively this is known as ***Weyl's criterion****.*

Proof.

1. $\implies$ 2.. 1. means 2. is true for the characteristic function of $[b, c]$, hence true for step functions (because everything is linear).
Recall that any $f \in C[0, 1])$ is the uniform limit of such functions.
Exercise: Choose one that is ϵ close to f and finish the proof.

2. $\implies$ 1.. Find $f_1, f_2 \in C([0, 1])$ such that $f_1(x) \leq \chi_{(b,c)}(x) \leq f_2(x)$ s.t.

$$\int f_1 - \chi_{(b,c)}) \leq \epsilon$$

Exercise, finish this proof.

2. $\implies$ 3.. Obvious.

3. $\implies$ 2.. Recall Fejér's theorem that $f \in C[0, 1])$ is the uniform limit of its fourier series. Pick an $n \in \mathbf{N}$ s.t.

$$f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon.$$

Exercise, finish this proof. ∎

Using this we can tackle more examples, e.g.:

1. $\{\alpha p_n\}$, n-th primes, $\alpha \in \mathbf{R} \setminus \mathbf{Q}$.
2. $\{\sqrt{n}\}$
3. $\{\log(n)\}$
4. $\{\log(p_n)\}$
5. $\{\beta^n\}$, $\beta \in \mathbf{R}$.
6. $\{\alpha a_n\}$, $a_n \in \mathbf{Z}$

1.5 Some applications of Weyl's criterion

Application 1. $\{n\alpha\}$ is uniformly distributed iff $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. Let

$$S_N(h, \alpha) = \sum_{n=0}^{N} e(h\alpha n)$$

then

$$S_N(h, \alpha) = \frac{e((N+1)h\alpha) - 1}{e(h\alpha) - 1}.$$

Then $e(h\alpha) - 1 \neq 0$ for all $h \in \mathbf{Z} \setminus \{0\}$ iff $\alpha \notin \mathbf{Q}$ so $S_N(h, \alpha) = O_{h,\alpha}(1)$ iff $\alpha \notin \mathbf{Q}$.
So $\lim_{N\to\infty} S_N(h, \alpha) = 0$, $\forall h \neq 0 \iff \alpha \notin \mathbf{Q}$. And $S_N(0, \alpha) = N + 1$ which

implies

$$\lim_{N\to\infty} \frac{1}{N} S_N(0,\alpha) = 1$$

so by Weyl's criterion $\{n\alpha\}$ is uniformly distributed if $\alpha \notin \mathbf{Q}$.

Digression.

Theorem 1.29 Vinogradov 1937. *$\{\alpha p_n\}$ is uniformly distributed iff $\alpha \notin \mathbf{Q}$.*

He actually showed weak Goldbach conjecture, Every (sufficiently large) *odd* integer is a sum of 3 primes. No bound by Vinogradov, Borozdin gave a large bound, Helfgott brought it down to reality.

Application 2.

$$\{\beta^n\}$$

Theorem 1.30 Koksma 1935. *For almost every (Lebesgue) $\beta \in \mathbf{R}_{>1}$, $\{\beta^n\}$ is equidistributed.*

Koksma was a Dutch student of Van der Corput.

Theorem 1.31 Weyl. *Let $|a_n| \to \infty$ be a sequence of distinct integers, the set of $\alpha \in \mathbf{R}$ such that αa_n is not uniformly distributed has Lebesgue measure 0.*

Application 3.

$$\{\log(n)\}$$

is not equidistributed.

Exercise 1.32 Prove this. Hint : assume it was

$$\#\{n < 1001N :\} = \#\{n < 1000N :\} + \#\{1000N \le n \le 1001N :\}$$

then

$$\log(n) - \log(100N + n_1) = \log(1000N) + \log(1 + n_1/1000N)$$

Application 4.

$$\{\sqrt{n}\}$$

Before this, lets introduce

Definition 1.33 Discrepancy. Let $a_n \in [0,1)$ be a sequence and $N \in \mathbf{Z}_{\ge 0}$ we define

$$D_N = \sup_{0\le b\le c\le 1} \left\{ \left| \frac{\#\{n < N : a_n \in [b,c]\}}{N} - (c-b) \right| \right\}.$$

◊

Some useful lemmas:

Lemma 1.34 *Let*

$$D_N^* = \sup_{0\le c\le 1} \left\{ \left| \frac{\#\{n < N : a_n \in [0,c]\}}{N} - c \right| \right\}$$

then

$$D_N^* \to 0 \iff D_N \to 0.$$

Proof. Exercise (hint squeeze). ■

Lemma 1.35 *a_n is uniformly distributed $\iff D_N \to 0 \iff D_N^* \to 0$.*

Proof. Ommitted/exercise, 2.1.1 of Kuipers and Niederreiter. ∎

For $\sqrt{n}$ now.

$$S_N(c) = \#\{n \leq N : 0 \leq \{\sqrt{n}\} \leq c\}.$$

Note 1.36 If $d = \lfloor \sqrt{n} \rfloor$ then

$$\sqrt{n} = d + \{\sqrt{n}\}$$

and $d \leq \sqrt{n} \leq d+1$ and $\#\{n : d \leq \sqrt{n} \leq d+1\} = 2d+2$.

Similarly

$$\#\{n : d \leq \sqrt{n} \leq d+\alpha\} = 2\alpha d + \alpha^2 + 1$$

then

$$S_N(\alpha) = \sum_{j=0}^{\lfloor \sqrt{N} \rfloor} \#\{n : j+\alpha \leq \sqrt{n} \leq j+\alpha\}$$

$$= \sum_{j=0}^{\lfloor \sqrt{N} \rfloor} 2\alpha d + \alpha^2 + O(1)$$

$$= 2\alpha \frac{\lfloor \sqrt{N} \rfloor (\lfloor \sqrt{N} \rfloor + 1)}{2} + O(\sqrt{N}) = \alpha N + O(\sqrt{N})$$

so

$$D_N^* = \sum_{0 \leq \alpha \leq 1} \left\{ \left| \frac{S_N(\alpha)}{N} - \alpha \right| \right\} = \sup_{0 \leq \alpha \leq 1} O_\alpha(1/\sqrt{N}) = O(1/\sqrt{N}).$$

Exercise 1.37 Make the dependence on α explicit and make the proof rigorous.

Application 5.
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Proposition 1.38

$$\{\log(p_n)\}$$

is not equidistributed.

Proof. Assume otherwise, let $k \in \mathbf{Z}_{>0}$.

$$I_k = \min\{n : p_n > e^k\}$$

$$I_{k-1/2} = \min\{n : p_n > e^{k-1/2}\}$$

$$\chi_{[0,1/2)} = \text{char. fn. of } [0, 1/2).$$

Consider

$$S_k = \sum_{n < I_k} \chi_{[0,1/2)}(\{\log(p_n)\})$$

$$S_{k-1/2} = \sum_{n < I_{k-1/2}} \chi_{[0,1/2]}(\{\log(p_n)\})$$

Note that $S_k = S_{k-1/2}$.
If uniformly distributed

$$\lim_{k\to\infty} \frac{S_k}{I_k} = \lim_{k\to\infty} \frac{S_{k-1/2}}{I_{k-1/2}} = L$$

$$L > 0 \implies I_k / I_{k-1/2} \to 1.$$

But the prime number theorem says

$$\pi(N) = \#\{p < N\} \sim \frac{N}{\log N}$$

so

$$I_k / I_{k-1/2} \sim \sqrt{e} \neq 1$$

so $L = 0$ but this cannot happen either.

$$S_{k-1/2} \geq \pi(e^{k-1/2}) - \pi(e^{k-1})$$

$$\geq e^{k-1} \frac{\sqrt{e} - 1}{k} > 0.$$

■

1.6 Weyl differencing and the Van der Corput inequality

Theorem 1.39 Weyl. *Let*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbf{R}[x]$$

then

$$\{P(n)\}$$

is uniformly distributed mod 1 iff there exists $n_0 \in \{1, \ldots, n\}$ s.t. $a_{n_0} \notin \mathbf{Q}$.

Lemma 1.40 Weyl differencing. *Let $y_n \in \mathbf{C}$, such that $y_n = 0$ if $n < 1$ or $n > N$ then we would like to bound*

$$\sum_m \sum_n y_n \overline{y}_m = \sum_{m,n=1}^{N} = S_N \overline{S}_N = |S_N|^2,$$

Weyl differencing *gives*

$$\sum_{m=1}^{N} \sum_{n=1}^{N} y_n \overline{y}_m = \underbrace{\sum_n |y_n|^2}_{\text{diagonal contribution}} + \underbrace{2\Re \left(\sum_{n=1}^{N-1} \sum_{m=1}^{N-n} y_{n+m} \overline{y}_m \right)}_{\text{off diagonal terms}}.$$

Proof. Idea: Use the transformation

$$n \mapsto n + m$$

$$m \mapsto m$$

$$\begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

switch orders

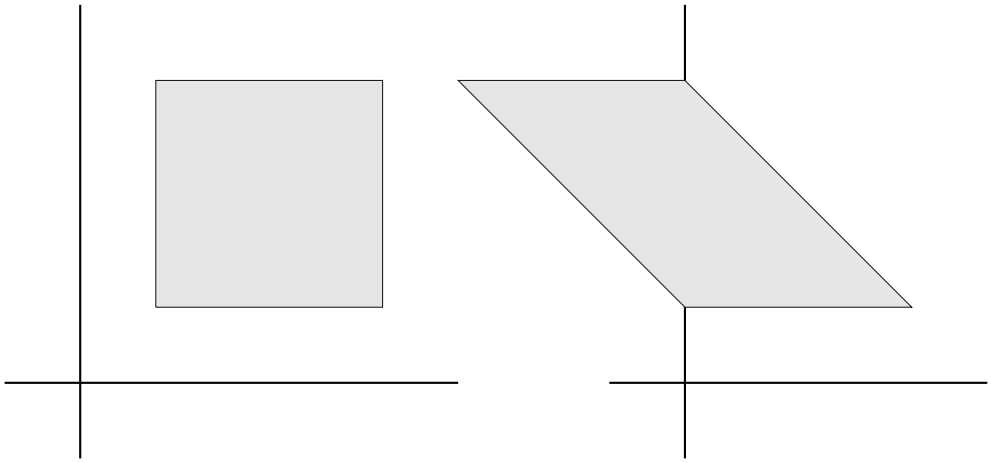

Figure 1.41: Regions

$$\sum_m \sum_n y_n \overline{y}_m = \sum_m \sum_n y_{n+m} \overline{y}_m$$

Note: if $n \geq 0$ then $m = 1, \ldots, N-n$, $n = 0, 1, \ldots, N-1$. If $n < 0$ then

$$m = 1-n, \ldots, N$$

$$n = 1-N, \ldots, -1$$

$$\sum_{m=1}^{N} |y_m|^2 + \sum_{n=1}^{N-1} \sum_{m=1}^{N-n} y_{n+m} \overline{y}_m + \sum_{n=-N+1}^{-1} \sum_{m=1-n}^{N} y_{m+n} \overline{y}_m$$

$$= \sum_{m=1}^{N} |y_m|^2 + 2\Re \left(\sum_{n=1}^{N-1} \sum_{m=1}^{N-n} y_{n+m} \overline{y}_m \right)$$

■

Theorem 1.42 Weyl. *Let $|x_n| \to \infty$ be a sequence of distinct integers, the set of $\alpha \in \mathbf{R}$ such that αx_n is not uniformly distributed has Lebesgue measure 0.*

Proof. Let $\alpha \in [0, 1)$, let $h \in \mathbf{Z} \setminus \{0\}$

$$S_h(N, \alpha) = \frac{1}{N} \sum_{n=1}^{N} e(h x_n \alpha)$$

the trick is now to bound $|S_h(N, \alpha)|^2$

$$N^2 |S_h(N, \alpha)|^2 = \sum_{n=1}^{N} 1 + 2\Re \left(\sum_{n=1}^{N-1} \sum_{m=1}^{N-n} e(h(x_{n+m} - x_m)\alpha) \right)$$

so

$$\int_0^1 |S_h(N, \alpha)|^2 \, \mathrm{d}\alpha = \frac{1}{N}.$$

Recall Fatou's lemma: Let $f_n > 0$ be sequence of positive measurable functions, then if

$$f(x) = \liminf f_n(x)$$

we have

$$\int f \leq \liminf \int f_n.$$

Take

$$f_n(\alpha) = \sum_{N=1}^{n} |S_h(N^2, \alpha)|^2$$

Fatou implies

$$\int_0^1 \sum_{n=1}^{\infty} |S_h(N^2, \alpha)|^2 \, \mathrm{d}\alpha \leq \sum_{N=1}^{\infty} \int_0^1 |S_h(N^2, \alpha)|^2 \, \mathrm{d}\alpha < \infty$$

so

$$\sum_{N=1}^{\infty} |S_h(N^2, \alpha)|^2$$

is finite for almost all α and for any $h \neq 0$. Then by Weyl's criterion we are done. ∎

Lemma 1.43 Van der Corput. *Let $y_n \in \mathbf{C}$, such that $y_n = 0$ if $n < 1$ or $n > N$ then*

$$\forall H \in \mathbf{Z}_{>0}$$

we have

$$\left|\sum_{n=1}^{N} y_n\right|^2 \leq \frac{N+H}{H+1} \sum_{n=1}^{N} |y_n|^2 + \frac{2(N+H)}{H+1} \sum_{h=1}^{H} \left(1 - \frac{h}{H+1}\right) \left|\sum_{n=1}^{N-h} y_{n+h} \overline{y}_n\right|.$$

Proof.

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Consider

$$(H+1)^2 \left|\sum_{n=1}^{N} y_n\right|^2$$

then

$$\begin{aligned}(H+1)^2 \left|\sum_{n=1}^{N} y_n\right|^2 &= \left|\sum_{h=0}^{H} \sum_{n=1}^{N} y + n\right|^2 \\ &= \left|\sum_{h=0}^{H} \sum_{n \in \mathbf{Z}} y_{n+h}\right|^2 \\ &= \left|\sum_{n \in \mathbf{Z}} \sum_{h=0}^{H} y_{n+h}\right|^2 \qquad (1.1)\end{aligned}$$

note n-sum is nonzero for at most $N + H$ terms $1 - H \leq n \leq N$ then Cauchy-Schwarz implies

$$\begin{aligned}&\leq (N+H) \sum_{n} \left|\sum_{h=0}^{H} y_{n-h}\right|^2 \\ &= (N+H) \sum_{n} \sum_{h_1=0}^{H} \sum_{h_2=0}^{H} y_{n+h_1} \overline{y_{n+h_2}} \qquad (1.2)\end{aligned}$$

let $m = n + h_2$, $l = h_1 - h_2$. Then

$$= (N+H) \sum_{-H \le l \le H} \sum_m \overline{y}_m y_{m-l} \underbrace{\left(\sum_{0 \le h_1, h_2 \le H} 1 \right)}_{= I(H,l)}$$

note $I(H,l) = H + 1 - |l|$, so

$$= (N+H) \sum_{-H \le l \le H} H + 1 - |l| \sum_m \overline{y}_m y_{m-l}$$

then if $l = 0$ we get the first term of the statement, $l \neq 0$ the other. ∎

What is the difference between this and Weyl differencing? When H is large, not so much, but we can take H small now, shifting the weighting around. We change the balance to make one part shorter and the other longer.

Theorem 1.44 Van der Corput differencing. *If for each $h \in \mathbf{Z}_{\ge 1}$ the sequence $b_h(n) = a_{n+h} - a_n$ then so is a_n.*

Proof. We'll use Weyl's criterion and the van der Corput lemma. Fix, $N \in \mathbf{Z}_{\ge 1}$ a $H \le N$. Then for any $k \in \mathbf{Z} \setminus \{0\}$

$$\left| \sum e(k a_n) \right|^2 \le \frac{N+H}{H+1} N + 2 \frac{N+H}{H+1} \sum \left(1 - \frac{h}{H+1} \right) \left| \sum_{n=1}^{N-h} e(b_h(n) k) \right|$$

$$\implies \lim_{N \to \infty} \frac{1}{N^2} \left| \sum_{n=1}^{N} e(k a_n) \right|^2 \le \frac{1}{H+1}$$

since H is arbitrary

$$\lim_{N \to \infty} \frac{1}{N^2} \left| \sum_{n=1}^{N} e(k a_n) \right|^2 = 0.$$

∎

Proof of Weyl's theorem. Let $\deg P = d$ then, for $d = 1$ we are done by Weyl's criterion, for $d \le D$ use van der Corput differencing. Note that for fixed h, $P(n+h) - P(n)$ is of lower degree. ∎

1.7 A different perspective (Ergodic)

Furstenberg (1981 book) gives a different proof that $\{n^2 \alpha\}$ is uniformly distributed.

Ergodic theory 101.

Definition 1.45 Ergodic measures. Let X be a locally compact space and H a non-compact group, $H \circlearrowright X$. μ a probability measure on X, H-invariant. We say that μ is an **ergodic measure** if any of the following equivalent conditions hold

1. $A \subseteq X$ and A is H-invariant ($hA = A$ for all $h \in H$). Then
$$\mu(A) = 1 \text{ or } \mu(A) = 0.$$
2. For f measurable μ almost everywhere H-invariant
$$f(hx) = f(x)$$

for almost all x then f is constant μ almost everywhere.

3. μ is an extreme point on the convex set of H-invariant probability measures.

◊

Definition 1.46 Uniquely ergodic actions. An action of a group H on a locally compact space X is **uniquely ergodic** if there is only one invariant probability measure on X. ◊

Example 1.47 $x \mapsto 5x$, or $x \mapsto \sqrt{2} + x$ on T^1 are ergodic, let's show that $x \mapsto \sqrt{2} + x$ is uniquely ergodic. □

Lemma 1.48 *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ then $T_\alpha : x \to x + \alpha$ is uniquely ergodic.*

Proof. Let μ be an invariant probability measure on $[0, 1]$ the nth Fourier coefficient

$$\hat{\mu}(n) = \int e(nx)\,\mathrm{d}\mu(x) = e(n\alpha)\int e(nx)\,\mathrm{d}\mu(x)$$

so $\hat{\mu}(n) = \delta(n)$. ■

Example 1.49 For $x \mapsto 5x$ $f = \sum a_n e(nx) = \sum a_n e(5nx)$ so $a_n = a_{5n}$

$$\infty > \|f\|^2 = \|a_n\|^2 \implies a_n = 0 \forall n \neq 0$$

$$\implies f \equiv C \text{ a.e.}$$

□

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Definition 1.50 Equidistribution. A sequence of probability measures μ_n on a locally compact space X is called μ-**equidistributed** if they converge to μ in the weak $*$ topology. I.e.

$$\forall f \in C_c(X), \int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu.$$

◊

Remark 1.51 If we have a sequence a_n these define a sequence of measures

$$\mu_N = \frac{1}{N}\sum_{n \leq N} \delta(x - a_n)$$

if these are equidistributed then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(a_n) = \int_X f(x)\,\mathrm{d}\mu(x).$$

Theorem 1.52 Birkhoff ergodic theorem. *(X, B, μ, T) a measure preserving system (as in the definition of ergodic) with μ a probability measure. Then for any $f \in L^1(X, \mu)$*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i(x) = \bar{f}(x)$$

for some $\bar{f} \in L^1(X, \mu)$ *moreover*

$$\int \bar{f} \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$$

and if T *is ergodic then* $\bar{f}(x) = \int f \, \mathrm{d}\mu$ *almost everywhere.*

Remark 1.53 This does not help us!
But the following does:

Theorem 1.54 *If* X *is compact and* $H \cong \mathbf{R}$ *is uniquely ergodic with the unique* H*-invariant measure* μ *then the statement of Birkhoff holds for every* $x \in X$*.*

Proof (sketch). For any x construct

$$\mu_L = \frac{1}{L} \int_0^L h_t x \, \mathrm{d}t$$

$$\mu_L \to \mu, \, L \to \infty$$

in the weak $*$ sense. ∎

Remark 1.55 Funny side remark (Benford's law). First digit of a set of observations are usually 1 (roughly 30%).
Why? Assumption, the process follows a power law, if b^n is the first digit of this is

$$\log_{10} b^n = n \log_{10} b$$

this is k iff $n \log_{10} b \in [\log_{10} k, \log_{10} k + 1]$. If these are equidistributed then the probability of k is

$$\int_{\log_{10} k}^{\log_{10} k+1} 1 \, \mathrm{d}x = \log(1 + 1/k).$$

Theorem 1.56 $\{n^2\alpha\}$ *is equidistributed for* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$*.*

Proof. Furstenberg

1. Construct a suitable dynamical system on T^2 (2-torus), for which a specific orbit gives the sequence $\{n^2\alpha\}$.
2. Will show that this is uniquely ergodic.

$$T \colon T^2 \to T^2$$

$$(x, y) \mapsto (x + \alpha, y + 2x + \alpha)$$

$$T_\alpha^n(x, y) = (x + n\alpha, y + 2nx + n^2\alpha)$$

in particular the orbit of $(0, 0)$ is

$$T^n(0, 0) = (n\alpha, n^2\alpha)$$

so if we show that $T^n(0, 0)$ is equidistributed we are done.
So we show that X, T is uniquely ergodic.

1. Lebesgue is ergodic.
2. Only ergodic measure

1.
$$f \in L^2(T^2, \mathrm{d}m)$$
T-invariant
$$f(x,y) = \sum_{m,n} a_{m,n} e(mx + ny)$$
T-invariance implies $a_{m,n} = e((m+n)\alpha) a_{m+2n,n}$ in particular
$$|a_{m,n}| = |a_{m+2n,n}|$$
so
$$a_{m,n} = 0$$
if $n \neq 0$. By Riemann-Lebesgue lemma. So for $n = 0$
$$\sum a_{m,0} e(mx)$$
T-invariance implies $a_{m,0} = e(m\alpha) a_{m,0}$ so $a_{m,0} = 0$ for all $m \neq 0$. So f is constant almost everywhere.

2. We use the following:
Claim:
$g : T^1 \to T^1$ measurable, let $T_g : T^2 \to T^2$
$$(x,y) \mapsto (x + \alpha, y + g(\alpha))$$
Then if the Lebesgue measure m is T_g-ergodic then it indeed is uniquely ergodic.
Proof: exercise.

■

2 Duke's theorem

Recall. Lecture 8 4/10/2018

1.2 1.9.

We will now prove this, not via Linnik's proof, but via an analogue of Weyl's criterion. Strategy:

1. Weyl's criterion $\rightsquigarrow$ bounding exponential sums.
2. To bound these sums we will use automorphic methods.

We will begin with 1.2.

So we will be working on S^2. Recall 1.28 which was for $[0,1)$. There we used exponentials, why? Because they are a dense, convenient basis, fourier theory.

To replace this on the sphere S^2 we use the *spherical harmonics*, these are homogeneous harmonic polynomials. Analogous with the S^1 case.
$$\left(\frac{x+iy}{|x+iy|}\right)^m = e(m\theta)$$

the same spherical harmonic construction works for S^n.

We will show

$$\frac{1}{\#\Omega_n}\sum_{x\in\Omega_n} P(x) \to 0$$

for $\deg P(x) > 0$. i.e.

$$\sum_{\alpha\in\mathbf{Z}^3, |\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right) = o(r_3(n)) \tag{2.1}$$

where

$$r_3(n) = \#\{a^2+b^2+c^2 = n : a,b,c \in \mathbf{Z}\}.$$

Connection to automorphic forms. θ functions, P spherical harmonics as before.

Definition 2.1

$$\theta_P(z) = \sum_{\alpha\in\mathbf{Z}^3} P(\alpha)e(|\alpha|^2 z)$$

for $z \in \mathbf{H}$ this converges.

$$\theta_P(z) = \sum_{n=0}^{\infty} r_3(n,P)e(nz)$$

$$r_3(n,P) = \sum_{|\alpha|^2=n} P(\alpha).$$

◊

Fact 2.2

1. *This is a modular form of weight* $\frac{3}{2} + \deg P$ *for* $\Gamma_0(4)$.
2. *It is a cusp form if* $\deg P > 0$.

Observation 2.3 If $\deg P \not\equiv 0 \pmod 2$ this function is 0.

To show (2.1):

1. Show that
$$r_3(n) \gg_\epsilon n^{1/2-\epsilon}$$
(Gauss-Siegel)
2. Show that
$$r_3(n,P) \ll_\delta n^{k/2-1/4-\delta}$$
for some $\delta > 0$.

Why?

$$\sum_{|\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right) = n^{-\deg(P)/2}\sum_{|\alpha|^2=n} P(\alpha)$$
$$= n^{-\deg(P)/2} r_3(n,P)$$

note:

$$\frac{k}{2} - \frac{1}{4} - \delta = \frac{1}{2} + \frac{\deg(P)}{2} - \delta.$$

So if $r_3(n,P) \ll n^{(1+\deg(P))/2-\delta}$ which implies

$$\sum_{|\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right) \ll n^{\frac{1}{2}-\delta}.$$

If we knew a half-integral weight Ramanujan conjecture:

$$f \in S_k(N),\ k \in \frac{1}{2} + \mathbf{Z}_{>0}$$

for squarefree n

$$a_f(n) = O(n^{(k-1)/2+\epsilon}).$$

Digression (θ-functions). Recall: Integral weight modular forms

$$\Gamma = \mathrm{SL}_2(\mathbf{Z}) = \Gamma(1)$$

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}$$

In general Γ congruence subgroup $\Gamma(N) \subseteq \Gamma \subseteq \Gamma(1)$ for some N.

$$f : \mathbf{H} \to \mathbf{C}$$

is called a modular form of weight k for Γ if f is holomorphic everywhere, including at the cusps.

$$f(\gamma z) = (cz+d)^k f(z)\ \forall \gamma \in \Gamma.$$

If $f(z) = \sum_{n=1}^{\infty} a_n q^n$ then f is cuspidal, the space of such is $S_k(\Gamma)$ where S stands for the German Spitzenform. Spitze means cusp, kinda like a pointy spit?

Example 2.4

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

$$E_k(z) = \sum_{(c,d)=1} \frac{1}{(cz+d)^{2k}}$$

□

Conjecture 2.5 Ramanujan (here theorem of Deligne). $f \in S_k(\Gamma)$

$$a_n = O_z(n^{(k-1)/2+\epsilon})\ \forall \epsilon > 0.$$

Classically

$$\widetilde{\theta}(z) = \sum_{m\in\mathbf{Z}} e^{i\pi m^2 z}$$

converges absolutely on $z \in \mathbf{H}$

$$\widetilde{\theta}(z+2) = \widetilde{\theta}(z)$$

$$\widetilde{\theta}(-1/z) = \sqrt{-iz}\widetilde{\theta}(z)$$

in general for $\gamma \in \Gamma_0(4)$

$$\widetilde{\theta}(\gamma z) = j(\gamma; z)\widetilde{\theta}(z)$$

where $j(\gamma;z) = \left(\frac{c}{d}\right)\epsilon_d^{-1}(cz+d)^{1/2}$ where $\epsilon_d = 1$ if $d \equiv 1 \pmod 4$, -1 if $d \equiv 3 \pmod 4$ sign of Gauss sum. $\left(\frac{c}{d}\right)$ is a sibling of Legendre form.

Lecture 9 11/10/2018

Recall: P spherical harmonic degree l. Aim to show:

$$\sum_{|x|^2=n} P(x) = o(r_3(n))$$

where

$$r_3(n) = \#\{a^2 + b^2 + c^2 = n : a, b, c \in \mathbf{Z}\}.$$

Took

$$\theta(z; P) = \sum_{n \in \mathbf{Z}} r(n; P) e(nz).$$

A modular form of weight $3/2 + l$ for $\Gamma_0(4)$. Cusp form if $l > 0$.

The strategy is then to show

$$r_3(n) \gg n^{1/2-\epsilon} \; \forall \epsilon > 0$$

$$r_3(n) \ll n^{k/2-1/4-\delta} \text{ for some } \delta > 0.$$

Definition 2.6 Half-integral weight modular forms. Let $N \equiv 0 \pmod 4$. A modular form of **half-integral** weight $k \in \frac{1}{2} + \mathbf{Z}_{\geq 0}$. For $\Gamma_0(N)$ is a holomorphic function on $\mathbf{H}$ s.t.

1.
$$f(\gamma z) = j(\gamma; z)^{2k} f(z)$$

2. f is holomorphic at the cusps

if $\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)$, $f(\gamma z) = \chi(d) j(\gamma; z)^{2k} f(z)$ gives the space $M_k(\Gamma_0(N), \chi)$. ◊

Where do these things come from?

A construction due to Schoenberg 1939, Pfetzer 1953, Shimura 1973 is as follows:

A an $n \times n$ positive definite integral matrix, $N \in \mathbf{Z}$ s.t. NA^{-1} is integral, P a spherical harmonic relative to A i.e. for P homogeneous of degree v.

$$\sum_{i,j} \tilde{a}_{ij} \frac{\partial^2 P}{\partial x_i x_j} = 0$$

$$[a_{ij}] = A^{-1}$$

Definition 2.7 Let $h \in \mathbf{Z}^n$, set

$$\tilde{\theta}_P(z, h, N) = \sum_{m \equiv h \pmod N} P(m) e\left(\frac{(m^{\mathrm{T}} A m) z}{2N^2}\right).$$

◊

Fact 2.8 Poisson summation.

$$\tilde{\theta}_P(\gamma z, h, N) = e\left(\frac{ab(h^{\mathrm{T}} A h)}{2N^2}\right) \left(\frac{\det A}{d}\right) \left(\frac{2c}{d}\right)^n \epsilon_d^{-n} (cz + d)^{k/2} \tilde{\theta}_P(z; ah, N)$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : b \equiv 0 \pmod 2, c \equiv 0 \pmod{2N}$$

$$k = n + 2v$$

Example 2.9 $n = 1, N = 1$ $P(m) = m^v$ $v = 0, 1$

$$\tilde{\theta}_P(z) = \sum_{m \in \mathbf{Z}} m^v e(m^2 z/2)$$

for $v = 0$ this is classical θ for $v = 1$ cusp form on $\Gamma_0(8)$ of weight $3/2$, we could also twist by a character mod 4. □

Example 2.10 $A = I_{n \times n}$ P spherical harmonic of degree v

$$\tilde{\theta}_P(z; 0, 1) = \sum_{m \in \mathbf{Z}^n} P(m) e(|m|^2 z/2)$$

set $z = 2z$

$$\theta_P(z) = \sum_{m \in \mathbf{Z}} P(m) e(|m|^2 z).$$

□

Note 2.11

$$\theta_P(z) \in S_k(\Gamma_0(4))$$

$$k = \frac{h}{2} + v, \ v > 0$$

$$\theta_P(z) \in M_{h/2}(\Gamma_0(4))$$

not cuspidal.

Example 2.12 $A = 4 \times 4$ integral positive definite

$$Q = Q_A(x), \ x^{\mathrm{T}} A x$$

$$r_Q(n) = \#\{x \in \mathbf{Z}^4 : Q(x) = n\}$$

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) e(nz) \in M_2(\Gamma_0(N))$$

for some $N = 4 \det(A)$. □

Half-integral weight Ramanujan conjecture (Metaplectic). Naively we would like to mimic the integral case and say

$$a_f(n) = O(n^{(k-1)/2+\epsilon})$$

where $f \in S_k(\Gamma)$.

But this is not true as stated here!:

Example 2.13 Let

$$\theta(z; \chi) = \sum_{m \in \mathbf{Z}} m \chi(m) e(m^2 z)$$

for odd χ this is a cusp form in $S_k(\Gamma, \chi)$. For $k = 3/2$, let $f = \theta$ and

$$a_f(m^2) \sim m\chi(m) \implies a_f(n) = O(\sqrt{n})$$

for n square. But $(k-1)/2 = 1/4$ but $O(\sqrt{n})$ is not $O(n^{1/4+\epsilon})$ for any $\epsilon \geq 1/4$.□

So we avoid these or stick to squarefree n.

Conjecture 2.14 Half-integral weight Ramanujan conjecture.

$$a_f(n) = O(n^{(k-1)/2+\epsilon})$$

for all n square-free $f \in S_k(\Gamma, \chi)$.

Why this exponent? For integral weight: Representation theory gives $a_f(m)$'s which correspond to Hecke eigenvalues which under Langlands are tempered for GL_N.

Digression:

Proposition 2.15 Hecke bound. $f \in S_k(\Gamma)$ *then* $a_f(n) = O(n^{k/2})$.

Proof. $y = \Im(z)$, $y(\gamma z) = |cz + d|^{-2} y$ so

$$F(z) = y^{k/2}|f(z)|$$

Invariant on Γ bounded at the cusps $|F(z)| < M$. So

$$\int_0^1 f(z)e(nx) = e^{-2\pi i n y} a_f(n)$$

$$\leq \int_0^1 y^{-k/2} M = O(y^{-k/2})$$

so $a_f(n) = O(y^{-k/2} e^{2\pi i n y})$ with $y = 1/n$. ∎

Another digression:

What is the distribution of rational points on S^2?

$$\{x_1^2 + x_2^2 + x_3^2 = 1 : x_i \in \mathbf{Q}\}$$

$$\iff r_3(n^2)$$

Proposition 2.16 Hurwitz. *A generating function for this is given by*

$$\sum_{n=1}^{\infty} \frac{r_3(n^2)}{n^s} = 6(1 - 2^{1-s}) \overbrace{\frac{\zeta(s)\zeta(1-s)}{L\left(s, \left(\frac{-1}{\cdot}\right)\right)}}^{=\sum \sigma(n)/n^s}$$

Exercise 2.17 Prove this!

David Fried's proof of Hurwitz on $r_3(n^2)$. Without proof Hurwitz stated

$$r_3(N^2) = 6P \prod_{q|Q} (q^a + 2q^{a-1} + \cdots + 2q + q)$$

where $N = 2^k PQ$ and each prime factor of P is $\equiv 1 \pmod 4$ and each prime factor of Q is $\equiv -1 \pmod 4$ and

$$q^a || Q$$

he suggested a proof along the lines of his published note on

$$r_5(N^2),$$

here is such a proof:

We denote $P = P(N)$, $Q = Q(N)$ We may suppose $k = 0$ since

$$r_3(4n) = r_3(n)$$

(each solution of $4n = x^2 + y^2 + z^2$ has x, y, z all even). So N is odd and each solution of $N^2 = x^2 + y^2 + z^2$ has two of x, y, z even. Hence

$$r_3(N^2) = \frac{3}{2} \sum_{x \text{ even}} r_2(N^2 - x^2) = \frac{2}{2} \sum_{a+b=2N, a, b \text{ odd}} r_2(ab)$$

now $r_2(n)$ is the number of Gaussian integers z with $z\bar{z} = n$. As $\mathbf{Z}[i]$ is a PID with 4 units $\pm 1, \pm i$, the function $\rho(n) = \frac{1}{4} r_2(n)$ is multiplicative!

$$\rho(ab) = \rho(a)\rho(b) \text{ if } a, b \text{ coprime}$$

clearly

$$r_3(N^2) = 6 \sum_{a+b=2N,\, a, b \text{ odd}} \rho(ab).$$

To evaluate $\rho(ab)$ we need spme standard functions,

$$\tau(n) = \#\text{divisors } d \text{ of } n, \ \sigma(n) = \sum d$$

the Möbius function $\mu(n)$. and

$$\square(n) = \frac{1}{2} r_1(n)$$

$$w(n) = \mu(P)|\mu(Q)|$$

The multiplicativity of ρ generalises as follows:

Lemma 2.18 *If a, b are odd with $g = \gcd(a, b)$ then*

$$\rho(ab) = \sum_{d|g} w(d) \rho\left(\frac{a}{d}\right) \rho\left(\frac{b}{d}\right)$$

Proof. Each $p = a^2 + b^2$ with $a > b > 0$ uniquely (Euler). Let $z(p) = a + bi$. For n odd $n = PQ$ and $r_2(n) = 0$ unless $Q = \square$. In which case the solutions of $n = z\bar{z}$ are

$$z = i^k \sqrt{Q} \prod_p z(p)^j \overline{z(p)}^{\alpha - j}$$

where $p^\alpha || P$, $k \in \mathbf{Z}/4$. Thus

$$\rho(n) = \tau(p)\square(Q).$$

When $\rho(ab) = 0$ every term on the right vanishes. For $Q(ab) \neq \square$ so

$$Q\left(\frac{a}{d}\right), Q\left(\frac{b}{d}\right)$$

cannot both be squares.
When $\rho(ab) \neq 0$ we have $Q(ab) = m^2$ for some$m$. There is exactly one square-free γ dividing m and g such that

$$Q\left(\frac{a}{\gamma}\right), Q\left(\frac{b}{\gamma}\right)$$

are squares. The term

$$w(d)\rho(a/d)\rho(b/d)$$

is nonzero just when $\alpha = \gamma\delta$ where δ is squarefree and $\delta | P(g)$ so the sum reduces to

$$\sum_{\delta} \mu(\sigma)\tau(P(a/\gamma\delta))\tau(P(b/\gamma\delta))$$

$$= \sum_{\delta} \mu(\delta)\tau(P(a)/\delta)\tau(P(b)/\delta)$$

but

$$\sum_{d|r,d|s} \mu(d)\tau(r/d)\tau(r/s) = \tau(rs)$$

by multiplicativity of τ this reduces to the case where $r = \pi^k, s = \pi^l$ for prime π but $1+k+l = (1+k)(1+l)-kl$. Here now $r = P(a), s = P(b), \gcd(r,s) = P(g)$. So the sum over δ equals

$$\tau(P(a)P(b)) = \tau(P(ab))\square(Q(ab)) = \rho(ab)$$

as desired. ∎

Using this we find

$$r_3(n^2) = 6\sum_{d|N} w(d) S\left(\frac{N}{d}\right)$$

where for odd n

$$S(n) = \sum_{a+b=2n,\, a,b \text{ odd}} \rho(a)\rho(b)$$

but

$$S(n) = \frac{1}{16}\#(2n = \underbrace{w^2 + x^2}_{\text{odd}} + \underbrace{y^2 + z^2}_{\text{odd}})$$

$$= \frac{1}{4}\#(2n = 4w^2 + 4x^2 + y^2 + z^2)$$

$$= \sigma(n)$$

by Jacobi.

So

$$r_3(N^2) = 6\sum_{d|N} w(d)\sigma\left(\frac{N}{d}\right)$$

$$= 6(w * \sigma)(N)$$

where $*$ denotes Dirichlet convolution.

Pasing to Dirichlet series

$$\sum_{N \text{ odd}} \frac{r_3(N^2)}{N^s} = 6\sum_{d \text{ odd}} \frac{w(d)}{d^s} \sum_{m \text{ odd}} \frac{\sigma(m)}{m^s}$$

$$= 6\underbrace{\prod_p (1-p^{-s}) \prod_q (1+q^{-s})}_{L(s,\left(\frac{-4}{\cdot}\right))^{-1}} \underbrace{\sum_{k \text{ odd}} \frac{1}{k^s}}_{\zeta_{\text{odd}}(s-1)} \underbrace{\sum_{l \text{ odd}} \frac{l}{l^s}}_{\zeta_{\text{odd}}(s-1)}$$

$$= 6\prod_p \sum_a \frac{p^a}{p^{as}} \prod_q \sum_a \frac{q^a + 2q^{a-1} + \cdots + 2q + 2}{q^{as}}$$

which gives

$$r_3(N^2) = 6 \prod_{p^a||N} p^a \prod_{q^a||N} (q^a + 2q^{a-1} + \cdots 2q + 2)$$

as in Hurwitz and

$$\sum_{n=1}^{\infty} \frac{r_3(n^2)}{n^s} = 6L(s, \left(\frac{-4}{\cdot}\right))^{-1} \zeta(s)\zeta_{\text{odd}}(s-1)$$

as in Duke.

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Corollary 2.19 *This shows*

$$r_3(n^2) \gg n.$$

Proof (sketch). Expand the formula, take $n = p$ an odd prime

$$\zeta(s)\zeta(1-s) = \sum \frac{\sigma(n)}{n^s}$$

(exercise)

$$L(s, \chi_{-4})^{-1} = \prod_p (1 - \frac{\chi_{-4}(p)}{p^s})$$

so

$$\sum \frac{r_3(n^2)}{n^s} = \sum \frac{\sigma(m_1)}{m_1^s} \prod (1 - \chi_{-4}(m_p)/p^s) 6(1 - 2^{1-s})$$

$$= \sum \frac{\sigma(m_1)}{m_1^s} \sum \mu(m_) \chi_{-4}(m_2)/m_2^s 6(1 - 2^{1-s})$$

so

$$r_3(p^2) = (\sigma(p) - \chi_{-4}(p))6 \geq 6p$$

■

The next big breakthrough came much later, showing something similar for $r_3(n, P)$, but now we hope to see cancellation.

Theorem 2.20 Shimura '71. *Let*

$$f \in S_{k/2}(N, \chi)$$

be a half integral weight cusp form, k odd, $4|N$.

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz),$$

assume f is a common eigenfunction for all T_{p^2}.
Let $\widetilde{\chi}(m) = \chi(m)\chi_{-4}(m)$ and $\lambda = (k-1)/2$. Set

$$\sum_{n=1}^{\infty} \frac{A(n)}{n^s} = L(s + 1 - \lambda, \widetilde{\chi}) \cdot \left(\sum_{m=1}^{\infty} \frac{a_f(m^2)}{m^s} \right)$$

$$F(z) = \sum_{n=1}^{\infty} A(n) e(nz).$$

Then

$$F(z) \in S_{2\lambda}(N_1, \chi^2).$$

Recall:

$$\theta_P(z) = \sum r_3(n;P)e(nz) \in S_{3/2+l}(\Gamma_0(4))$$

is a weight $3/2 + l$ form if $l = \deg P$.

Shimura gives us

$$F_P(z) \in S_{2l+2}(\Gamma_0(2))$$

then we get

Corollary 2.21

$$r_3(n^2, P) = \sum_{d|n} A_P(d)\mu\left(\frac{n}{d}\right)\chi_{-4}\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^l$$

hence if $A_P(d) \ll d^{l+1-\delta}$ then we are done.

The Hecke bound implies $A_P(d) \ll d^{l+1}$.

Detour: bounding Fourier coefficients. There are various approaches, such as Poincaré series, Kloosterman sums, and bound these, to get a bound on the Fourier coefficient.

If $f \in S_{k_1}(N_1)$ we will write a spanning set for $S_{k_1}(N_1)$, then bound fourier coefficients on each of these guys, which will suffice.

Poincaré series:

If we were just looking for a modular form we might try Eisenstein series

$$E_k(z) = \sum_{c,d} \frac{1}{(cz+d)^{2k}}$$

these are not cuspidal. But if we twist this a little bit

$$P_m(z;k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma;z)^{-2k} e(m\gamma z).$$

Where $\Gamma_\infty = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right\}$.

$$j(\gamma;z) = \left(\frac{c}{d}\right)\epsilon_d^{-1}(cz+d)^{1/2}$$

they converge absolutely for $k > 2$.

Some properties:

Lemma 2.22

$$P_m(\tau z;k) = (cz+d)^{2k} P_m(z;k)$$

Proof. Exercise, j satisifes a cocycle relation

$$j(\gamma;\tau z)j(\tau;z) = j(\gamma\tau;z).$$

■

Proposition 2.23

$$P_m(z) \in S_k(\Gamma)$$

Proof. To follow. ■

Lemma 2.24 *Let* $f(z) = \sum a_f(n)e(nz)$

$$\left\langle P_m(z;k), f \right\rangle_{\text{Pet}} = \frac{\overline{a}_f(m)}{(4\pi m)^{k-1}}\Gamma(k-1).$$

Proof.

$$\begin{aligned}
\left\langle P_m(z;k), f \right\rangle_{\text{Pet}} &= \int_{\Gamma\backslash\mathbf{H}} P_m(z;k)\overline{f}(z)y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_{\Gamma\backslash\mathbf{H}} \sum_{\gamma\in\pm\Gamma_\infty\backslash\Gamma} j(\gamma;z)^{-2k} e(m\gamma z)\overline{f}(z)y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \sum_{\gamma\in\pm\Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash\mathbf{H}} j(\gamma;z)^{-2k} e(m\gamma z)\overline{f}(z)y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_{\Gamma_\infty\backslash\mathbf{H}} e(mz)\overline{f}(z)y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_0^1 \int_0^\infty e(mz)\overline{f}(z)y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \overline{a}_f(m) \int_0^\infty e(2miy)y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \frac{\overline{a}_f(m)}{(4\pi m)^{k-1}}\Gamma(k-1)
\end{aligned}$$

■

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Corollary 2.25 *$P_m(z;k)$ span $S_k(\Gamma)$ as m varies through integers.*
Proof. If $f \perp P_m(z,k)$ for all m then $f = 0$. ■

Observation 2.26 The corollary implies that bounding fourier coefficients of $f \in S_k(\Gamma)$ is equivalent to bounding fourier coefficients of $P_m(z,k)$.

Proposition 2.27

$$\langle P_m(z,k), P_n(z,k)\rangle = \left(\frac{m}{n}\right)^{(k-1)/2}\left(\delta_{m,n} + \frac{2\pi}{i^k}\sum_{c\equiv 0 \pmod N} J_{k-1}\left(\frac{2\pi\sqrt{mn}}{c}\right)\frac{K(m,n;c)}{c}\right)$$

where

J_k is the Bessel function of the first kind

N s.t. $\Gamma = \Gamma_0(N)$

$$K(m,n;c) = \sum_{d\in(\mathbf{Z}/c)^\times}\left(\frac{c}{d}\right)^{2k}\epsilon_d^{-2k}e\left(\frac{m\overline{d}+nd}{c}\right)$$

$$\delta_{m,n} = \begin{cases} 1 & m = n, \\ 0 & m \neq n \end{cases}.$$

Proof.

$$\int_0^1 P_m(z,k)e(-nz)\,\mathrm{d}z = \int_0^1 \left(\sum_{\pm\Gamma_\infty\backslash\Gamma} j(\gamma;z)^{-2k}e(m\gamma z)\right)e(-nz)\,\mathrm{d}z$$

$$= \int_0^1 \underbrace{j(1;z)^{-2k} e((m-n)z)}_{\delta_{m,n}} \,\mathrm{d}z + \text{rest}$$

$$= \int_0^1 j(1;z)^{-2k} e((m-n)) \,\mathrm{d}z + \text{rest}$$

Note:

$$e(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} z) = e(z).$$

So

$$\int_0^1 \sum_{\Gamma_\infty \backslash \Gamma} j(\gamma;z)^{-2k} e(m\gamma z) \,\mathrm{d}z$$

$$= \int_0^1 sum_{\Gamma_\infty \backslash \Gamma / \Gamma_\infty} \sum_{\gamma_\infty \in \Gamma_\infty} j(\gamma\gamma_\infty z)^{-2k} e(m\gamma\gamma_\infty z) e(-nz) \,\mathrm{d}z$$

$$\int_0^1 \sum_{\Gamma_\infty \backslash \Gamma / \Gamma_\infty} \sum_{\alpha \in \mathbf{Z}} j(\gamma; z+\alpha)^{-2k} e(m\gamma(z+\alpha)) e(-nz) \,\mathrm{d}z$$

$$\int_0^1 \sum_{\Gamma_\infty \backslash \Gamma / \Gamma_\infty} j(\gamma;z)^{-2k} e(m\gamma z - nz) \,\mathrm{d}z$$

Note:

$$m\gamma z = m\frac{az+b}{cz+d} = m\left(\frac{a}{c} - \frac{1}{c(cz+d)}\right)$$

$$\pm\Gamma_\infty \backslash \Gamma / \Gamma_\infty = \{(a,d,c) : c > 0,\ a,d \in (\mathbf{Z}/c)^\times,\ ad \equiv 1 \pmod{c}\}$$

So our main integral is

$$\sum_{c>0,\, d \in \mathbf{Z}/c^\times} \epsilon_d^{-2k} \left(\frac{c}{d}\right)^{2k} \int_{-\infty}^{\infty} \frac{1}{(cz+d)^k} e\left(m\frac{a}{c} - \frac{m}{c(cz+d)} - nz\right) \mathrm{d}z$$

$$\sum_{c>0,\, d \in \mathbf{Z}/c^\times,\, ad \equiv 1 \pmod{c}} \frac{e((ma+nd)/c)}{c^k} \epsilon_d^{-2k} \left(\frac{c}{d}\right)^{2k} \int_{-\infty}^{\infty} \frac{1}{z^k} e\left(-\left(zn + \frac{m}{c^2 z}\right)\right) \mathrm{d}z$$

Then use the following integral representation

$$\int_{-\infty+iA}^{\infty+iA} w^{-k} e^{-(\mu_1 w + \mu_2 m^{-1})} \,\mathrm{d}w$$

$$= 2\pi \left(\frac{\mu_1}{\mu_2}\right)^{(k-1)/2} e^{ik\pi/2} J_{k-1}(4\pi\sqrt{\mu_1\mu_2})$$

■

To recap:

$$P_m(z) = \sum_{n=1}^{\infty} \hat{P}_m(n) e(nz)$$

$$\hat{P}_m(n) = \left(\frac{n}{m}\right)^{(k-1)/2} \left\{ \delta_{m,n} + \frac{2\pi}{i^k} \sum_{c \equiv 0 \pmod{N},\, c>0} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{K(m,n;c)}{c} \right\}$$

Digression: Let $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, $k = 12$ then $S_k(\Gamma) = \langle \Delta \rangle$

$$\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e(nz).$$

As $\Delta = \eta^{24}$ we have $\tau(n) \in \mathbf{Z}$.
As

$$P_m(z;12) \in S_{12}(1) = \langle \Delta \rangle$$
$$\implies P_m(z;12) = \kappa(m)\Delta$$

to calculate $\kappa(m)$

$$\langle P_m(z), \Delta \rangle = \frac{\overline{a}_m}{(4\pi m)^{k-1}} \Gamma(k-1) = \frac{2\pi \overline{\tau(m)}}{(4\pi m)^{k-1}} \Gamma(k-1) = \frac{2\pi 10!}{(4\pi m)^{11}}$$

so

$$P_m(z) = \frac{2\pi 10!}{(4\pi m)^{11}} \frac{\Delta(z)}{\|\Delta\|^2_{\mathrm{Pet}}} \tau(m).$$

Another consequence:

Theorem 2.28 Petersson trace formula. *For any $m, n \geq 1$*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f \text{ o.n. basis for } S_k(\Gamma)} \overline{a}_f(m) a_f(n) = \delta_{m,n} + \frac{2\pi}{i^k} \sum_{c \equiv 0 \pmod N, c>0} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{K(m,n;c)}{c}.$$

Proof. Let $P_m(z) = \sum_f \left\langle P_m(z), f \right\rangle f$ then

$$\langle P_m(z), P_n(z) \rangle = \sum_f \overline{a_f(m)} a_f(n) \frac{\Gamma(k-1)^2}{(4\pi m)^{k-1}(4\pi n)^{k-1}}$$

on the other hand

$$\langle P_m(z), P_n(z) \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \left(\frac{m}{n}\right)^{(k-1)/2} \{\cdots\}.$$

■

Lecture 12 23/10/2018

Recall Shimura gave us

$$\sum_{n=1}^{\infty} \frac{r_3(n^2, P)}{n^s} = \frac{\sum_{n=1}^{\infty} \frac{A_P(n)}{n^s}}{L(s-l, \chi_{-4})}$$

where the numerators is the L-function of a modular form of weight $2l+2$ where $l = \deg P$.

We want to show $A_P(n) \ll n^{1+l-\delta}$ for some $\delta > 0$.

Idea: Poincaré series span cusp forms, so its enough to bound Fourier coefficients of these.

We know

$$\hat{P}_m(n) = \left(\frac{n}{m}\right)^{(k-1)/2} \left(\delta_{m,n} + 2\pi i^{-k} \sum_{c \equiv 0 \pmod N, c>0} \overbrace{\frac{K(m,n,c)}{c}}^{\sum_{x \pmod{*c}} e((xm+\overline{x}n)/c)} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \right).$$

Try: Input 1:

$$J_{k-1}(x) \ll \min\{x^{k-1}, 1/\sqrt{x}\}.$$

Trivial input:

$$|K(m,n;c)| < c.$$

For

$$\begin{aligned} c \gg \sqrt{mn} \rightsquigarrow J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) &\sim \left(\frac{\sqrt{mn}}{c}\right)^{k-1} \\ &< \left(\frac{n}{m}\right)^{(k-1)/2} \sum_{c \gg \sqrt{nm}} \frac{1}{c^{k-1}} (mn)^{(k-1)/2} \\ &< n^{k-1} \frac{1}{\sqrt{mn}^{k-2}} = O(n^{k-1-(k-2)/2}) \end{aligned}$$

for

$$\begin{aligned} c \ll \sqrt{mn} \rightsquigarrow J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) &\sim \frac{\sqrt{c}}{(mn)^{1/4}} \\ &< \left(\frac{n}{m}\right)^{(k-1)/2} \sum_{c \ll \sqrt{nm}} \frac{\sqrt{c}}{(mn)^{1/4}} = O(n^{k/2}). \end{aligned}$$

Note $k/2 = l+1$: So we are just short, we have $O(n^{l+1})$ rather than $O(n^{l+1-\delta})$ for all $\delta > 0$, just by using a stupid bound for $K(m,n;c)$.

More major input (Weil bound):

$$\begin{aligned} |K(m, n\chi)| &\ll_\epsilon c^{1/2+\epsilon} \\ \implies |\hat{P}_m(n)| &= O(n^{k/2-1/4+\epsilon}) \end{aligned}$$

so

$$\forall \epsilon < \frac{1}{4}$$

this does the job.

So we have finished proving Linnik for n^2, it wasn't too bad, the major inputs were the Shimura correspondence.

When we move to the general case things will be totally different.

Remark 2.29 Weil bound comes from the Riemann hypothesis for curves over finite fields.

Lemma 2.30 *Let p be an odd prime $p \nmid mn$.*

$$\sum_{x \pmod{^*p}, xy=1} e\left(\frac{xm + \overline{y}n}{p}\right) = \sum_{x^2 - n\overline{m} = y^2 \pmod{p}} e\left(\frac{2mx}{p}\right)$$

Proof. Reparameterise $xy = 1$, $x = a + b$, $y = a - b$. Exercise. ∎

Lemma 2.31

$$\sum_{x \pmod{p}} \left(\frac{x^2 - n\overline{m}}{p}\right) e\left(\frac{2mx}{p}\right) = \sum_{x^2 - \overline{m}n y^2} e\left(\frac{2xm}{p}\right).$$

Proof.

$$\sum_{x \pmod{p}} \left(\frac{x^2 - n\overline{m}}{p}\right) e\left(\frac{2mx}{p}\right) = \sum_{x \pmod{p}} \left(\left(\frac{x^2 - n\overline{m}}{p}\right) + 1\right) e\left(\frac{2mx}{p}\right).$$

■

Note 2.32 LHS is calculating a Fourier coefficient. i.e.

$$h_{m,n}(\alpha) = \left(\frac{\alpha^2 - \overline{m}n}{p}\right)$$

$$\text{LHS} = \hat{h}(2m)$$

the above two lemmas imply

$$K(m, n; p)$$

is the Fourier coefficient $\hat{h}_{m,n}(m)$.

So

$$h(\mathbf{Q}(\sqrt{u^2 - 4\overline{m}n})) \sim L\left(1, \left(\frac{u^2 - 4\overline{m}n}{\cdot}\right)\right)$$

Selberg trace formula $\leftrightarrow_{\text{Fourier transform}}$ Petersson trace formula.

Remark 2.33 In the non-square case we want $|r_3(n, P)| \ll n^{k/2-1/4-\delta}$.

This is because of the subtlety of the Shimura correspondence.

More precisely for n squarefree $n^{k/2-1/4-\delta}$

$$\theta(z; P) \text{ weight } \frac{3}{2} + \deg P = k$$

$$r_3(n; P) \gg n^{1/2-\delta}$$

want

$$r_3(n; P) \ll n^{k/2-1/4-\delta}$$

$$\sum_{|\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right) = |n|^{-\deg P/2} r_(n, P)$$

$$r_3(n, P) < n^{k/2-1/4-\delta} = n^{3/4+\deg P/2-1/4-\delta}$$

for n^2

$$r_3(n^2) \gg n$$

have Shimura

$$\theta(z; P) \rightsquigarrow \text{weight } k$$

$$A_p(n^2)$$

are fourier coefficients of a weight $2 \deg P + 2$ form.

Back to $r_3(n, P)$. n-squarefree, we don't have Shimura, but we still have the Fourier expansion of $P_m(z)$.

Strategy: We will first exploit a certain form of these Kloosterman sums for Δ squarefree, Salié sums. Major detail

$$\sum_c \frac{K(m, n; c)}{c}$$

will bound these not individually, but by showing that the angles of these cancel. We show other equidistribution results from equidistribution of Kloosterman sums essentially.

Lecture 13 25/10/2018

Main issue.

$$\sum_c \frac{K(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

saw that bounding each individual $K(m,n;c)$ falls short of what is needed.

Detour (Salié sums). Recall for $k = \frac{3}{2} + l$ the sums we are getting are

$$K_k(m,n;c) = \sum_{d \pmod{*c}} \epsilon_d^{-2k} \left(\frac{c}{d}\right) e\left(\frac{md + n\overline{d}}{d}\right)$$

where

$$\left(\frac{c}{d}\right) \text{ Kronecker symbol}$$

$$\epsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ i & d \equiv -1 \pmod 4 \end{cases}$$

It will turn out that these sums can be calculated in elementary terms. Analogy

$$K_k(m,n;c) \leftrightarrow K_k$$

for $k \in \frac{1}{2} + \mathbf{Z}$, $J_k(x)$ is expressible in terms of elementary functions, e.g.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin(x)$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\sin(x)}{x^{3/2}} - \frac{\cos(x)}{\sqrt{x}}\right)$$

Calculation: Reductions:

Lemma 2.34 *Let $c = rq$ with $(r,q) = 1$ and $4 \mid r$,*

$$K_k(m,n;c) = K_{k-q+1}(m\overline{q}, n\overline{q}; r) S(m\overline{r}, n\overline{r}; q)$$

where

$$S(m,n;q) = \sum_{x \pmod{*q}} \left(\frac{x}{q}\right) e\left(\frac{mx + n\overline{x}}{q}\right)$$

is the Salié sum.

Exercise 2.35 Prove this.

Lemma 2.36 *q prime*

1. $$S(m,n;q) = \left(\frac{m}{n}\right) S(q, mn; q)$$

2. $$S(1,m;q) = 0$$

   unless $\left(\frac{m}{q}\right) = 1$.

3. $$S(1,n^2;q) = \underbrace{\epsilon_q \sqrt{q}}_{G(q)} \sum_{x^2 \equiv 1 \pmod q} e\left(\frac{2xn}{q}\right)$$

Proof.

1. $$x = \overline{m}y$$

2. $$x = m\overline{y}$$

3. $$S(1, n^2; q) = \sum_{x \pmod{{}^*q}} \left(\frac{x}{q}\right) e\left(\frac{x + n^2\overline{x}}{q}\right)$$

   recall DFT

$$f(u) = \frac{1}{q} \sum_{\alpha \pmod{q}} \hat{f}(\alpha) e\left(\frac{\alpha u}{q}\right)$$

$$\hat{f}(\alpha) = \sum_{u \pmod{q}} f(u) e\left(\frac{-\alpha u}{q}\right)$$

$$= \sum_{q \pmod{q}} \sum_{x \pmod{{}^*q}} \left(\frac{x}{q}\right) e\left(\frac{x + u^2\overline{x}}{q}\right) e\left(\frac{-\alpha u}{q}\right)$$

$$= \sum_{u \pmod{q}, x \pmod{{}^*q}} \left(\frac{x}{q}\right) e\left(\frac{x}{q}\right) e\left(\frac{y^2\overline{x} - \alpha u}{q}\right)$$

$$= \sum_{x \pmod{{}^*q}} \left(\frac{x}{q}\right) e\left(\frac{x(1 - \overline{4}x^2)}{q}\right) \underbrace{\sum_{u \pmod{q}} e\left(\frac{\overline{x}(u - 2\alpha x)^2}{q}\right)}_{\left(\frac{x}{q}\right) G(q)}$$

$$G(q) \sum_{x \pmod{{}^*q}} e\left(\frac{x(1 - \overline{4}\alpha^2)}{q}\right)$$

we need some more results to conclude!

■

Lemma 2.37 *Let*

$$c_q(r) = \sum_{x \pmod{{}^*q}} e\left(\frac{rx}{q}\right)$$

then

$$c_q(r) = \sum_{d | gcd(q,r)} \mu\left(\frac{q}{d}\right) d.$$

Proof.

$$c_q(r) = \sum_{x \pmod{{}^*q}} e\left(\frac{xr}{q}\right)$$

$$= \sum_{d|q} \sum_{a \pmod{{}^*q/d}} e\left(\frac{ar}{q/d}\right)$$

$$= \sum_{d|q} c_{q/d}(r)$$

$$= \sum_{d|q} c_d(r)$$

let

$$F(q) = \sum_{d|q} c_d(r)$$

then by Möbius inversion

$$c_q(r) = \sum_{d|q} \mu\left(\frac{d}{q}\right) F(d)$$

$$F(d) = \sum_{x \pmod d} \left(\frac{xr}{d}\right) = \begin{cases} d & \text{if } d|r, \\ 0 & \text{otw} \end{cases}$$

■

Note 2.38

$$c_q(r) = \mu\left(\frac{q}{\gcd(q,r)}\right) \frac{\phi(q)}{\phi(q/\gcd(q,r))}$$

This lemma implies that in the proof above we have

$$G(q)c_q(q - \overline{4}\alpha^2) = G(q) \sum_{d|\gcd(q,(1-\overline{4}\alpha^2))} \mu\left(\frac{q}{d}\right) d$$

so putting everything together

$$f(u) = \frac{1}{q} \sum_{\alpha \pmod q} \left(G(q) \sum_{d|\gcd(q,(1-\overline{4}\alpha^2))} \mu\left(\frac{q}{d}\right) d \right) e\left(\frac{\alpha u}{q}\right)$$

$$= \frac{G(q)}{q} \sum_{d|q} \sum_{\alpha \pmod q, \overline{4}\alpha^2 - 1 \equiv 0 \pmod d} de\left(\frac{\alpha u}{q}\right)$$

Lemma 2.39

$$\sum_{\alpha \pmod q, \overline{4}\alpha^2 - 1 \equiv 0 \pmod d} e\left(\frac{\alpha u}{q}\right) = 0 \text{ if } d \neq q$$

Proof. If $d \neq q$ then $de = q$, $e \neq 1$, $\alpha = \alpha_1 + d\alpha_2$ so

$$\sum_{\alpha} = \sum_{\alpha_1, d, \overline{4}\alpha_1^2 = 1 \pmod d} e\left(\frac{\alpha_1 u}{q}\right) \sum_{\alpha_2} e\left(\frac{\alpha_2 u}{q}\right) = 0.$$

■

This finishes the proof for Salié sums.

Theorem 2.40 Major theorem (Iwaniec '87). *Use this calculation + Petersson formula + genus Let $f \in S_k(N)$ of weight $2k \geq 1$ and $2k$ odd. Then for n squarefree and $\forall \epsilon > 0$:*

$$a_f(n) \ll_\epsilon n^{k/2 - 1/4 - 1/28 + \epsilon}.$$

Corollary 2.41

$$r_3(n, P) \ll n^{k/2 - 1/4 - 1/28 + \epsilon}$$

for all $\deg(P) > 0$.

Final input: Lower bound (Gauss, Siegel): Gauss showed

$$r_3(n) = \frac{24}{w(d)} h(d) \left(1 - \left(\frac{d}{2}\right)\right)$$

where $h(d)$ is the class number $\mathbf{Q}(\sqrt{-n})$ where $d = \text{disc}(\mathbf{Q}-n)$ and $w(d)$ is the number of roots of unity in this field.

Exercise 2.42 Show this

Question 2.43 How large is $h(d)$? □

Euler, Siegel

Theorem 2.44

$$h(D) \gg_\epsilon |D|^{1/2-\epsilon}$$
$$L(q, \chi_D) \gg_\epsilon |D|^{-\epsilon}$$

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Last time $f \in S_k(\Gamma), k \in \frac{1}{2} + \mathbf{Z}$ for n squarefree Iwaniec implies $|a_f(n)| \ll_\epsilon n^{k/2-1/4-1/28+\epsilon}$. This gives $r_3(n; P) \ll n^{k/2-1/4-\delta}$.

Finding the argument P = spherical harmonic, $\deg(P) = l$ and $h = l + 3/2$.

$$r_3(n; P) = \sum_{|\alpha|^2=n} P(\alpha).$$

Today:
$P \equiv 1$ main contribution.
$P \not\equiv 1$ error.

$P \equiv 1$.

$$r_3(n) = \sum_{|\alpha|^2=n} 1$$

(with no local obstructions, i.e. n squarefree and $n \not\equiv 7 \pmod 8$).

Theorem 2.45 Gauss.

$$r_3(n) = \frac{24h(\tilde{n})}{w(\tilde{n})} \left(1 - \left(\frac{\tilde{n}}{2}\right)\right)$$

where

$$\tilde{n} = \text{disc}(\mathbf{Q}(\sqrt{-n}))$$
$$w(\tilde{n}) = \#\textit{roots of 1 in } \mathbf{Q}(\sqrt{-n})$$

Note 2.46 $\tilde{n} \sim n$ as we are squarefree so off by at most a factor of 4. Then $r_3(n) \sim h(\tilde{n})$

Exercise 2.47 Prove the theorem.

We now wish to demonstrate a lower bound on $h(\tilde{n})$.

Theorem 2.48 Siegel '35. $\forall \epsilon > 0$ *have*

$$L(1, \chi_d) \gg_\epsilon |d|^{-\epsilon}.$$

Corollary 2.49

$$h(\tilde{n}) \gg_\epsilon n^{1/2-\epsilon}.$$

Exercise 2.50 Prove the corollary (Analytic class number formula)

Note 2.51 The constant in Siegel's theorem is ineffective, meaning one cannot actually calculate it (without proving there are no Landau-Siegel zeroes).

Finally Siegel + Iwaniec implies

$$\frac{1}{\sqrt{n}}\#\{\alpha \in \mathbf{Z}^3 : |\alpha|^2 = n\}$$

gets equidistributed on S^3 as $n \to \infty$.

Recall this above was problem 1.

Problem 2. $\Gamma = \mathrm{SL}_2(\mathbf{Z})$

$$\frac{1}{\#\Lambda_d} \sum_{z_Q \in \Lambda_d} f(z_Q) \to \int_{\Gamma\backslash\mathbf{H}} f(z)\,\mathrm{d}\mu(z)$$

where for $Q = ax^2 + bxy + cy^2$, positive definite.

$$z_Q = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \in \mathbf{H}.$$

Theorem 2.52 Duke '88. *Problem 2 is true.*

Duke's proof.

1. What to replace the exponential sums, i.e. the P's?
2. Follow the same strategy and bound the nontrivial sums from above ($P \not\equiv 1$), trivial sums from below ($P \equiv 1$).

Recall $[0,1) \rightsquigarrow P = e(nx)$, $S^2 \rightsquigarrow P$ spherical harmonics, $\Gamma\backslash\mathbf{H} \rightsquigarrow$ automorphic forms $E(z,s)$ or Maass cusp forms.

Digression $\Delta_{\mathrm{hyp}} \circlearrowright \Gamma\backslash\mathbf{H}$ fourier transform on $\mathbf{R}$ then $f(x) = \int_{i\mathbf{R}} \hat{f}(\alpha)e(\alpha x)\,\mathrm{d}x$

$$S^1 \implies f(x) = \sum_{n\in\mathbf{Z}} \hat{f}(n)e(nx)$$

i.e.

$$L^2(\Gamma\backslash\mathbf{H}) = \oint_{\mathbf{R}} E(z, \frac{1}{2} + it)\,\mathrm{d}\mu(t)$$
$$\bigoplus \sum_{\lambda} \phi_\lambda$$

ϕ_λ a Maass cusp form of eigenvalue λ.

Detour: Maass forms and Eisenstein series

Definition 2.53 Maass form. $f : \mathbf{H} \to \mathbf{C}$ is a **Maass form** if $\forall \gamma \in \Gamma = \mathrm{SL}_2(\mathbf{Z})$.

$$f(\gamma z) = f(z)$$

f is an eigenfunction for

$$\Delta_{\mathrm{hyp}} = -y^2\left(\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\right)$$

$$f(z) = O(y^N)$$

for some N.
Cusp form if

$$\int_0^1 f(x+iy)\,\mathrm{d}x = 0.$$

◊

Example 2.54 Eisenstein series.

$$E(z,s) = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{im}(\gamma z)^s$$

□

Fact 2.55

1. $E(z,s)$ *converges for* $\Re(s) \gg 1$.
2. *Has analytic continuation to* $\mathbf{C}$ *in the* s*-variable.*
3. *Has a simple pole at* $s=1$ *which is a constant* $\mathrm{Res}_{s=1} E(z,s) = \frac{1}{2}$.
4. $E(z,s) = E(z,1-s)$.
5. $E(z,s) \sim y^\sigma$ *where* $\sigma = \max\{\Re(s), \Re(1-s)\}$.

Back on track

$$\sum_{|\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right)$$

$$\sum_{z_Q \in \Lambda_d} 1$$

$$\sum_{z_Q \in \Lambda_d} E(z_Q, \frac{1}{2}+it)$$

$$\sum_{z_Q \in \Lambda_d} \phi_\lambda(z_Q)$$

Lemma 2.56

$$\zeta(2s) \sum_{z_Q \in \Lambda_d} E(z_Q, s)$$

$$\left(\frac{|d|}{4}\right)^{\frac{s}{2}} \underbrace{\zeta_{\mathbf{Q}(\sqrt{d})}(s)}_{=\zeta(s)L(s,\chi_d)}$$

Proof.

$$\sum_{z_Q} \zeta(2s) E(z_Q, s) = \sum_{z_Q} \sum_{(u,v) \in \mathbf{Z}^2 \setminus (0,0)} \frac{y_Q^2}{|uz_Q+v|^2 s}$$

Λ_d the class group of binary quadratic forms of discriminant d.

$$|uz_Q+v|^2 = |u\left(\frac{-b+\sqrt{d}}{2a}\right)+v|^2$$

$$= \frac{av^2 - uvb + cv^2}{a}$$

so we get in the above

$$= \left(\frac{|d|}{4a}\right)^{s/2} \sum_{Q \in \Lambda_d} \frac{1}{Q(u,v)^s}.$$

■

This gives

1.
$$\sum_{z_Q} 1 \sum |d|^{1/2} L(1, \chi_d) \gg_\epsilon |d|^{1/2-\epsilon}$$

by Siegel.

2.
$$\sum_{z_Q} E(z_Q, \frac{1}{2} + it) \sim \frac{|\zeta(\frac{1}{2} + it) L(\frac{1}{2} + it, \chi_d)|}{|\zeta(1 + 2it)|} |d|^{\frac{1}{4}}$$

want
$$L(\frac{1}{2} + it, \chi_d) \ll |d|^{\frac{1}{4}-8}.$$

We also need:

Theorem 2.57 de la Vallée Poussin.

$$\zeta(1 + 2it) \gg \textit{something} > 0$$

$$|\zeta(1 + 2it)| \gg \log(2 + 1 + 1)^{-1}$$

Phragmén-Lindelöf principle gives convexity bound on the d-aspect

$$L(\frac{1}{2} + it, \chi_d) \ll_\epsilon |d|^{\frac{1}{4}+\epsilon}.$$

Subconvexity bound

Theorem 2.58 Burgess.

$$L(\frac{1}{2} + it, \chi_d) \ll_\epsilon |d|^{\frac{1}{16}+\epsilon}$$

What about
$$\sum \phi_\lambda(z_Q)?$$

Harder even!

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Last time
$$\frac{1}{|\Lambda_d|} \sum_{z_Q \in \Lambda_d} f(z_Q) \to \int_{\Gamma \backslash \mathbf{H}} \mathrm{d}\mu(z)$$

Siegel's theorem gives a lower bound

$$\frac{1}{|\Lambda_d|} \sum_{z_Q \in \Lambda_d} 1$$

Subconvexity bound on $L(\frac{1}{2}+it,\chi_d)$ and non-vanishing $\zeta(1+2it)$.

$$\frac{1}{|\Lambda_d|}\sum_{z_Q\in\Lambda_d} E(z_Q,\frac{1}{2}+it)$$

is

$$\frac{1}{|\Lambda_d|}\sum_{z_Q\in\Lambda_d} \phi(z_Q)$$

a cusp form?

Waldspurger's formula. Roughly Waldspurger says

$$\frac{|a_{|D|}|}{\langle f,f\rangle}=\frac{(k-\frac{3}{2})!}{\pi}\frac{|D|^{k-1}}{\langle g,g\rangle}L(g\otimes\chi_D,k-\frac{1}{2})$$

$f\in S_k(\Gamma_0(4)$ half integral weight, $f\mapsto g$ Shimura lift, a_n fourier coeffs of F.

Remark 2.59 Shimura lifting takes weight k to weight $2k-1$ so $k-\frac{1}{2}$ is the center of the critical strip of

$$L(g\otimes\chi_D,s).$$

$$L(g\otimes,\chi_d,s)=\sum\frac{b_n\chi_D(n)}{n^s}.$$

$$\sum\phi(z_Q)\rightsquigarrow a_{\theta_\phi}(|D|)$$

Katok-Sarnak, compare with

$$P\left(\frac{\alpha}{|\alpha|}\right)\rightsquigarrow\sum r_3(P,n)e(nz)=\theta_P(z)$$

Then "all we need to do" is to show that

Subconvexity bound.

$$L(g\otimes\chi_D,k-\frac{1}{2})\ll|D|^{\frac{1}{2}-\delta}$$

this would be enough

Why? Recall

$$\sum_{|\alpha|^s=n}P(\alpha)\rightsquigarrow\theta_P(z)$$

$$\sum_{|\alpha|=n}P\left(\frac{\alpha}{|\alpha|}\right)=\frac{1}{n^{\deg P/2}}r_3(P;n)$$

$\theta_P(z)$ of weight $\deg P+\frac{3}{2}$.

Want

$$\frac{|a_{|D|}|^2}{|D|^{\deg P}}\ll|D|^{1-s}$$

$$g=\text{weight } 2k-1-2\deg P+2$$

$$\sim\frac{|D|^{k-1}}{|D|^{k-\frac{3}{2}}}L(g\otimes\chi_D,k-\frac{1}{2})$$

$$\sim |D|^{\frac{1}{2}} L(g \otimes \chi_D, k - \frac{1}{2})$$

final touch:

$$\sum_{z_Q \in \Lambda_d} 1 \gg |d|^{1/2-\epsilon}$$

so if we can get the subconvex bound

$$\frac{|a_{|D|}|^2}{|D|^{k-3/2}} \ll |D|^{1-\delta}$$

it is enough.

Conjecture 2.60 Lindelöf. *In the D-aspect*

$$L(g \otimes \chi_D, \frac{1}{2}) \ll |D|^{\epsilon}$$

$$L(g, \frac{1}{2} + it) \ll (1+t)^{\epsilon}$$

This follows from GRH.

Lemma 2.61 Convexity bound. *For all $\epsilon > 0$:*

$$\zeta(\frac{1}{2} + it) = O_\epsilon((1+|t|)^{1/4+\epsilon}).$$

Proof (sketch).

$$|\zeta(1+\epsilon+it)| \leq \zeta(1+\epsilon) \leq c_{\epsilon>0}.$$

Functional equation + Stirling gives

$$|\zeta(-\epsilon - it)| = O_\epsilon(|t|^{1/2+\epsilon})$$

$$|\zeta(\sigma + it)| \sim |\zeta(1-\sigma-it)| \left|\frac{t}{2\pi}\right|^{\frac{1}{2}-\sigma}.$$

$$f(s) = \zeta(s)\zeta(1-s)$$

+ Phragmén-Lindelöf, bound it on $a = -\epsilon, b = 1+\epsilon$:

$$|f(s)| \leq |t|^{1/2+\delta}.$$

■

3 Linnik's theorem

We will mostly follow Ellenberg, Michel, Venkatesh. Linnik's ergodic method and the distribution of integer points on spheres.

3.1 Linnik's theorem

Problem.

Lecture 16 6/11/2018

Distribution of

$$\mathscr{H}_d = \{(x,y,z) \in \mathbf{Z}^3 : x^2 + y^2 + z^2 = d\}$$

as $d \to \infty$.

Theorem 3.1 Linnik. *As $d \to \infty$ squarefree such that $d \not\equiv 7 \pmod 8$, $d \equiv \pm 1 \pmod 5$*

$$\left\{ \frac{x}{\sqrt{d}} : x \in \mathscr{H}_d \right\}$$

is equidistributed w.r.t. the Lebesgue measure

$$\operatorname{red}_\infty : \mathscr{H}_d \to S^2$$

$$\frac{|\operatorname{red}_\infty^{-1}(\Omega)|}{|\mathscr{H}_d|} \to \operatorname{Vol}(\Omega)$$

in the sense that

$$\frac{|\operatorname{red}_\infty^{-1}(\Omega)|}{|\mathscr{H}_d|} = \operatorname{Vol}(\Omega)(1 + o(1))$$

for any Ω a reasonable subset of S^2.

The $\pm 1 \pmod 5$ condition comes from a splitting condition in $\mathbf{Q}(\sqrt{d})$ that we need to find an appropriate group action.

We will rather focus on the following variant

Theorem 3.2 Linnik variant.

$$\overline{\mathscr{H}}_d = \{(x, y, x) \pmod q : (x, y, z) \in \mathscr{H}_d\}$$

with q fixed > 0 and $(q, 30) = 1$ (non-essential). As $d \to \infty$, d squarefree such that $d \not\equiv 7 \pmod 8$ and $d \equiv \pm 1 \pmod 5$. $\overline{\mathscr{H}}_d(q)$ becomes equidistributed in $\mathscr{H}_d(q) = \{(x, y, z) \in (\mathbf{Z}/q)^3 : x^2 + y^2 + z^2 = d\}$ w.r.t. counting measure, explicitly

$$\frac{|\operatorname{red}_q^{-1}(x)|}{|\mathscr{H}_d|} = \frac{1}{|\mathscr{H}_d(q)|}(1 + o(1))$$

as $d \to \infty$.

Actually we will prove something a little strange.

Definition 3.3 Deviation. ∞, let $\rho > 0$, $x \in S^2$ and consider the cap $\Omega(x, \rho)$.

$$\operatorname{dev}_d(\Omega) = \frac{1}{\operatorname{Vol}(\Omega)} \frac{|\operatorname{red}_\infty^{-1}(\Omega)|}{|\mathscr{H}_d|} - 1$$

(we want to show this is $o(1)$).
q, let $\bar{x} \in \mathscr{H}_d(q)$ then

$$\operatorname{dev}_d(\bar{x}) = \frac{|\mathscr{H}_d(q)||\operatorname{red}_q^{-1}(\bar{x})|}{|\mathscr{H}_d|} - 1.$$

◊

Theorem 3.4 *Fix $\delta, \eta > 0$ then for every*

$$\rho \geq d^{-1/4+\eta}$$

$$\frac{1}{q} \geq d^{-1/4+\eta}$$

$(q, 30) = 1$, *then as* $d \to \infty$

$$\mu_{S^2}(x : \mathrm{dev}_d(\Omega(x, \rho)) \geq \delta) \to 0$$

$$\frac{|\{\bar{x} \in \mathscr{H}_d(q) : \mathrm{dev}_d(\bar{x}) \geq \delta\}|}{|\mathscr{H}_d(q)|} \to 0.$$

How? Sketch of the argument:
Fix $\delta, \eta > 0$,

$$B_\delta = \{\bar{x} \in \mathscr{H}_q(d) : \mathrm{dev}_d(\bar{x}) > \delta\}$$

and assume that

$$|B_\delta| \geq \eta |\mathscr{H}_d(q)|.$$

Inputs

1. There is an action of $H(d)$ = class group of $\mathbf{Q}(\sqrt{-d})$ on $\mathscr{H}_d(q)/\mathrm{SO}_3(\mathbf{Z})$. As $d \equiv \pm 1 \pmod 5$ so $(5) = \mathfrak{p}\bar{\mathfrak{p}}$ in $\mathbf{Q}(\sqrt{-d})$: This gives us a well defined dynamical system by considering the action

   $$\mathscr{H}_d(q) \ni x \xrightarrow{\mathfrak{p}} x_1 \to x_2 \to x_2 \to \cdots$$

   and in reverse

   $$x_{-2} \to x_{-1} \to x \xrightarrow{\mathfrak{p}} x_1 \to x_2 \to x_2 \to \cdots$$

   call this chain γ_X and for any $l > 0 \in \mathbf{Z}$

   $$\gamma_X^{(l)} = [x_{-l}, x_{-l+1}, \ldots, x_{-1}, x, x_1, \ldots, x_l]$$

2. Consider the average on

   $$B_\delta = \gamma_X^{(l)} \cap B_\delta.$$

   On one hand the expected size of $\gamma_X^{(l)} \cap B_\delta$ is

   $$\frac{1}{|\mathscr{H}_d|} \sum_{x \in \mathscr{H}_d} |\gamma_X^{(l)} \cap B_\delta|$$

   on the other hand the expectation is

   $$\frac{(2l+1)}{|\mathscr{H}_d|} \mathrm{red}_q^{-1}(B_\delta).$$

   i.e.

   $$\frac{1}{|\mathscr{H}_d|} \sum_{x \in \mathscr{H}_d} \frac{|\gamma_X^{(l)} \cap B_\delta|}{2l+1} = \frac{|\mathrm{red}_q^{-1}(B_\delta)|}{|\mathscr{H}_d|}$$

3. Since we assume that the deviation set is large ($\mathrm{dev}_d(x) > \delta$) this implies on average

   $$\frac{|\gamma_X^{(l)} \cap B_\delta|}{2l+1}$$

   is large, gives a lower bound for this.

4. Then we count $\gamma_X^{(l)}$ in another way, action of $H(d)$ on $\mathscr{H}_d$ and on $\mathscr{H}_d(q)$. action of $\mathfrak{p}$ induces a graph structure on $\mathscr{H}_d(q)$. Then $\gamma_X^{(l)}$ is a path on $\mathscr{H}_d(q)$ which is non-backtracking. Count these: Inputs: for $l \sim \log(d)$ the number of non-backtracking paths for which $|\gamma_X^{(l)} \cap B_\delta|/(2l+1)$ is large is $\gg d^{1/2+\epsilon}$. Count the total number of non-backtracking paths of length $2l+1$ for which

$$\left| \frac{|\gamma_X^{(l)} \cap B_\delta|}{2l+1} - \frac{|B_\delta|}{|\mathscr{H}_d(q)|} \right| > \delta\eta/2$$

There can not be too many of these. This will give a contradiction. Will take some work and depend on many things.

Lecture 17 8/11/2018

Class group actions.

$$\mathscr{H}_d = \{(x,y,z) \in \mathbf{Z}^2 : x^2+y^2+z^2 = d\}$$

Aim:

1. $\widetilde{\mathscr{H}_d^*}$ is a principal homogeneous space for $H(d) = \mathrm{Cl}(\mathbf{Q}(\sqrt{-d}))$.
2. Describe the action explicitly.

Definition 3.5 Principal homogeneous spaces. Let G be a group, $X \neq \emptyset$ a set. X is a **principal homogeneous space** for G if

$$G \circlearrowright X$$

in a transitive, free manner. X is also called a G-**torsor**. ◊

Remark 3.6 Of G is non-abelian we have notions of a left-torsor and right-torsor.

Example 3.7

- G is a G-torsor
- $G = V$ a n-dimensional vector space then $\mathbf{A}^n$ is a G-torsor
- $G = \mu_n$ then $X(2) = \{x \in \mathbf{C} : x^n = 2\}$

□

Note 3.8

$$\mathrm{SO}_3(\mathbf{Z}) \circlearrowright \mathscr{H}_d$$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{H}_d,\ \gamma \in \mathrm{SO}_3(\mathbf{Z}) \rightsquigarrow \gamma v \in \mathscr{H}_d.$$

Exercise 3.9

1. $$\#\,\mathrm{SO}_3(\mathbf{Z}) = 24$$
2. $\mathrm{SO}_3(\mathbf{Z})$ has a subgroup $\mathrm{SO}_3^+(\mathbf{Z})$ of index 2 acting via even permutations.

Definition 3.10

$$\widetilde{\mathscr{H}_d} = \mathrm{SO}_3(\mathbf{Z})\backslash\mathscr{H}_d$$
$$\widetilde{\mathscr{H}_d}(q) = \mathrm{SO}_3(\mathbf{Z})\backslash\mathscr{H}_d(q)$$
$$\widetilde{S^2} = \mathrm{SO}_3(\mathbf{Z})\backslash S^2$$
$$\widetilde{\mathscr{H}_d^*} = \begin{cases} \mathrm{SO}_3^+(\mathbf{Z})\backslash\mathscr{H}_d, & d \equiv 1,2 \pmod 4 \\ \mathrm{SO}_3(\mathbf{Z})\backslash\mathscr{H}_d, & d \equiv 3 \pmod 4 \end{cases}$$

◊

For all practical purposes we can basically assume that $d \equiv 3 \pmod 4$.

Proposition 3.11 *$\widetilde{\mathscr{H}_d^*}$ is a principal homogeneous space for $H(d)$. Idea: Venkov 1922: Use quaternion algebras.*

Digression: Hamilton Quaternions.

$$B = \{u + ia + jb + kc : u, a, b, c \in \mathbf{R}\}$$
$$\bar{x} = u - ia - jb - kc$$
$$N(x) = x\bar{x} = u^2 + a^2 + b^2 + c^2$$
$$\mathrm{Tr}(x) = x + \bar{x} = 2u$$
$$B^{(0)} = \{x \in B : \mathrm{Tr}(x) = 0\}$$

Note
$$N(x) = a^2 + b^2 + c^2 \text{ if } x \in B^{(0)}.$$

$$B^\times = \text{units},\ B^1 = \{x \in B : N(x) = 1\}$$
$$PB^\times = B^\times / Z(B^\times)$$

Remark 3.12
$$B^\times \circlearrowright B^{(0)}$$
via conjugation.

Exercise 3.13
$$1 \to Z(B^\times) \to B^\times \to PB^\times \simeq \mathrm{SO}_3(\mathbf{Q}) \to 1$$

Definition 3.14 Hurwitz quaternions.

$$B(\mathbf{Z}) = \mathbf{Z}[i, j, \frac{i + j + k}{2}]$$

◊

The Lipschitz quaternions are

$$\mathbf{Z}[i, j, k].$$

1. $B(\mathbf{Z})$ is a maximal order
2. $B(\mathbf{Z})$ is a Euclidean domain
3. Lipschitz quaternions are not a Euclidean domain

Exercise 3.15

1. Show these properties.
2. Show $\mathrm{Cl}(\mathbf{Z}[i,j,k]) = 2$.

2 implies $B(\mathbf{Z})$ is a PID

$$(\mathbf{Q}^3, a^2+b^2+c^2) \simeq (B^{(0)}, N)$$

as metric spaces

The image of

$$B^\times(\mathbf{Z}) \to \mathrm{SO}_3(\mathbf{Z})$$

is $\mathrm{SO}_3^+(\mathbf{Z})$.

The action of $H(d)$. Start with $x = (a,b,c) \in \mathscr{H}_d$. This gives rise to an embedding

$$\iota_x : \mathbf{Q}(\sqrt{-d}) \hookrightarrow B$$
$$\sqrt{-d} \mapsto x + ai + bj + ck$$

note $x^2 = -d$.

Let $K = \mathbf{Q}(\sqrt{-d})$ ι_x is integral ion the sense that if

$$O_x = B(\mathbf{Z}) \cap \mathbf{Q}[x]$$

then

$$\iota^{-1}(O_x) = O_K.$$

Exercise 3.16 Prove this

$I \in H(d) \rightsquigarrow y = y(x,I)$
Consider $B(\mathbf{Z})\iota_x(I) = B(\mathbf{Z})q$ for some $q \in B(\mathbf{Q})$. Since $B(\mathbf{Z})$ is a PID.
Then $y = qxq^{-1} \in \mathscr{H}_d$

Note 3.17

1. $$y^2 = -d$$
2. If we take λI for $\lambda \in K^\times$, $y(x,\lambda I) = y(x,I)$ so we get a well defined construction $H(d)$.
3. q is defined up to conjugation by $B(\mathbf{Z})^\times$.

To sum up:
For any $x \in \mathscr{H}_d$ we have a well defined map

$$\mathrm{Cl}(d) \to \widetilde{\mathscr{H}_d^\times}$$
$$I \mapsto y(x,I).$$

Claim 3.18 *This map makes $\widetilde{\mathscr{H}_d^\times}$ a principal homogeneous space over* $\mathrm{Cl}(d)$.

Let $x, y \in \mathscr{H}_d$.
Construct an ideal $\Lambda_{x\mapsto y}$ s.t. $y(x, \Lambda_{x\mapsto y}) = y$.

Definition 3.19

$$\Lambda_{x\mapsto y} = \{\lambda \in B(\mathbf{Z}) : x\lambda = \lambda y\}.$$

◊

$x, y \in \mathscr{H}_d$ Witt's theorem implies there exists $q \in PB^\times(\mathbf{Q}) = \mathrm{SO}_3(\mathbf{Q})$ s.t.

- $$y = q^{-1}xq$$
- $$\Lambda_{x\mapsto y} = B(\mathbf{Z}) \cap \mathbf{Q}[x]q$$

Note 3.20 $\Lambda_{x\mapsto y}$ is locally free and rank 1

$$\implies [\Lambda_{x\mapsto y}] \in \mathrm{Pic}(\mathcal{O}_x) \simeq \mathrm{Pic}(\mathcal{O}_K) \simeq \mathrm{Cl}(d).$$

Lecture 18 13/11/2018

Summary $\widetilde{\mathscr{H}_d}^*$ is a principal homogeneous space for $\mathrm{Cl}(d)$, the class group of $\mathbf{Q}(\sqrt{-d})$.

How? $x \in \mathscr{H}_d$ defined

$$\mathbf{Q}(\sqrt{-d}) \overset{i}{\hookrightarrow} B(\mathbf{Q})$$

$$\sqrt{-d} \mapsto x$$

any $I \in \mathrm{Cl}(d) \rightsquigarrow B(\mathbf{Z})i(I) = B(\mathbf{Z})q$ for some q.

$$B(\mathbf{Z}) = \{x + ai + bj + ck : a, b, c, d \in \mathbf{Z}, \text{ or all } \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}\}$$

$$y = q^{-1}xq$$

$$Ix = y$$

$$[\Lambda_{x\mapsto y}]$$

inverse of this construction

$$\mathrm{SO}(3, \mathbf{Q}) \simeq PB^{\times}(\mathbf{Q})$$

Proposition 3.21 *$\widetilde{\mathscr{H}_d}^*$ is a principal homogeneous space for* $\mathrm{Cl}(d)$.

$$\widetilde{\mathscr{H}_d} = \widetilde{\mathscr{H}_d}^*$$

if $d \equiv 3 \pmod 4$. *The stabiliser of any point in $\widetilde{\mathscr{H}_d}$ is the order 2 subgroup generated by* $\mathfrak{p}|(2)$.

$$\implies |\mathscr{H}_d| = 24|h(d)| \text{ if } d \equiv 3 \pmod 4$$

$$|\mathscr{H}_d| = 12|h(d)| \text{ if } d \equiv 1, 2 \pmod 4$$

Proof. Requires checking everything works as intended, it is local. ∎

Explicit realization.

$$P \in \mathrm{Cl}(d)$$

$$\rightsquigarrow M_P \in \mathrm{SO}(3)$$

$$x \in \mathscr{H}_d \implies Px = M_P x$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Let

$$P\bar{P} = (5)$$

because of the assumption on d.

P action, let $x \in \mathscr{H}_d$.

$$N(P) = 5 \rightsquigarrow q \in B(\mathbf{Z})$$

$$N(q) = 5 \implies 5 \in \{1 \pm 2i, 1 \pm 2j, 1 \pm 2k\} B^{\times}(\mathbf{Z})$$

two of these will give you the action of $P, \bar{P}$.

Example 3.22 q will act by $q^{-1}xq$ acting on $B^{(0)}$, enough to check the action on i, j, k. Say

$$q = 1 + 2i, \bar{q} = (1 - 2i)/5$$

$$\bar{q}jq = \frac{(1-2i)}{5} j(1+2i)$$

$$= \frac{j - 2k}{5} 1 + 2i$$

$$j - 2k - 2k - 2j????????$$

□

Basic construction. Let $\mathscr{A}_5 = \{M_1^{\pm}, M_2^{\pm}, M_3^{\pm}\}$ starting with a solution $x \in \mathscr{H}_d$ one gets two matrices $\omega_1, \omega_{-1} \in \mathscr{A}_5$

$$x_{-3} \xleftarrow{\omega_{-3}} x_{-2} \xleftarrow{\omega_{-2}} x_{-1} \xleftarrow{\omega_{-1}} x \xrightarrow{\omega_1} x_1 \xrightarrow{\omega_2} x_2 \xrightarrow{\omega_3} x_3$$

where w_1 comes from P and w_{-1} from $\bar{P}$.

This construction gives a well-defined (up to flip) path starting with a solution $x \in \mathscr{H}_d$.

Remark 3.23 These chains are periodic.

Back to proof. Idea: if $x \in \mathscr{H}_d$ then x and these paths are well defined mod p.

We are aiming for

Proposition 3.24 *Let* $\Sigma(d, l, q)$ *denote*

$$\#\{(x, x') \in \mathscr{H}_d \times \mathscr{H}_d : \gamma_x^{(l)} = \gamma_{x'}^{(l)} \text{ in } \mathscr{H}_d\}$$

where

$$\gamma_x^{(l)} \in \mathscr{H}_d(q)$$

is the chain cut at $[-l, l]$ *and then reduced mod* q. *Then*

$$\sigma(l, d, q) \ll_{\epsilon} |\mathscr{H}_d| + d^{\epsilon}(1 + \frac{d}{q^2 5^{2l}})$$

Proof. Next time, Linnik's lemma $\#\{xy = d\} \leq d^{\epsilon}$. One more input. ■

Lecture 19 27/11/2018

In $\mathbf{Q}(\sqrt{d})$ we have $(5) = \mathfrak{p}\bar{\mathfrak{p}}$

$$x \in \mathscr{H}_d = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = d\}$$

there are 6 matrices $\mathcal{A} = \{A^{\pm}, B^{\pm}, C^{\pm}\}$, $A, B, C \in \mathrm{GL}_2(\mathbf{Z})$, reduction of these (mod q) $\circlearrowleft \mathscr{H}_d(q)$.

This induces a graph structure on $\mathscr{H}_d(q)$, (multiple edges between vertices are allowed).

$$\mathscr{H}_d \to \mathscr{H}_d(q)$$
$$[\ldots, x_{-1}, x_0, x_1, \ldots] \to \text{path in graph}$$

for $l \in \mathbf{N}$,

$$\gamma_x^{(l)} \subseteq \mathscr{H}_d(q).$$

Linnik's basic lemma. Generalisation of the following

$$r(d) = \sum_{a|d} 1 \leq d^{\epsilon}\ \forall \epsilon.$$

Proposition 3.25 Linnik. *Fix d, let $c \in \mathbf{Z}$, s.t. $|c| < d$. Then $\forall \epsilon > 0$,*

$$\#\{(x_1, x_2) \in \mathscr{H}_d^2 : x_1 \cdot x_2 = e\} \ll_{\epsilon} d^{\epsilon}.$$

Proof. Short detour on representations of quadratic forms by other quadratic forms. ■

Definition 3.26 Let $(Q, \mathbf{Z}^m)$, $(R, \mathbf{Z}^n)$, $m \geq n$ be nondegenerate quadratic forms. Then Q is said to represent R if there exists a $\mathbf{Z}$-linear map

$$\iota : \mathbf{Z}^n \to \mathbf{Z}^m$$

s.t.

$$Q(\iota(x)) = R(x)\ \forall x \in \mathbf{Z}^n.$$

◊

Example 3.27

$$Q(x, y) = x \cdot y$$
$$R(x) = dx^2$$

then let

$$\iota : \mathbf{Z} \to \mathbf{Z}^2$$
$$x \mapsto (ax, bx) \text{ where } ab = d$$

then

$$Q(\iota(x)) = R(x).$$

□

Let $x_1, x_2 \in \mathscr{H}_d$ s.t. $x_1 \cdot x_2 = e$.

$$x_1 = (a_1, b_1, c_1)\ x_2 = (a_2, b_2, c_2)$$

consider

$$\iota : \mathbf{Z}^2 \to \mathbf{Z}^3$$
$$(u, v) \mapsto (ua_1 + va_2, ub_1 + ub_2, uc_1 + vc_2) = ux_1 + vx_2$$

let $Q(a, b, c) = a^2 + b^2 + c^2$.

$$\begin{aligned} Q(\iota(u, v)) &= (ua_1 + va_1)^2 + (ub_1 + vb_2)^2 + (uc_1 + vc_2)^2 \\ &= du^2 + 2euv + dv^2 \\ &= R_{d,e}(u, v). \end{aligned}$$

So the number of representations of R by $Q/\mathrm{SO}_3(\mathbf{Z})$ bounds number of solutions $x_1 x_2 = e$.

Theorem 3.28 Gordon Pell 1949. *Let*

$$r(a,b,c) = \{\text{reps. of } ax^2+bxy+cy^2 \text{ by } x^2+y^2+z^2\}$$

then

$$r(a,b,c) = 24\cdot 2^{\nu}\prod_{p|2(b^2-4ac)} r_p(a,b,c)$$

where

$$\nu = \#\text{primes dividing } b^2-4ac$$

$$r_p(a,b,c) = O(1) \text{ unless } p^2|\gcd(a,b,c)$$

for us $r(d,2e,d)$, $b^2-4ac = 4(e^2-d^2)$ *and*

$$r(d,2e,d) \ll_\epsilon d^\epsilon.$$

Exercise 3.29 Show this (recall d is a fundamental discriminant).

Lemma 3.30 Shadowing lemma. *Let*

$$[x_{-l},\dots,x_0,x_1,\dots,x_l],[x'_{-l},\dots,x'_0,x'_1,\dots,x'_l]$$

be two marked paths in $\mathscr{H}_d$. *Then these paths have the same reduction mod* q *iff* $x_0 = \pm x'_0 \pmod{q5^l}$.

Theorem 3.31

$$\Sigma(d,l,q) = \#\{(x,x') \in \mathscr{H}_d^2 : \gamma_x^{(l)} = \gamma_{x'}^{(l)} \in \mathscr{H}_d(q)\}$$

$$\implies \Sigma(d,l,q) \ll_\epsilon \#\mathscr{H}_d + d^\epsilon(1+\frac{d}{q^2 5^{2l}})\ \forall \epsilon > 0.$$

Proof.

$$\gamma_x^{(l)} = \gamma_{x'}^{(l)} \implies x = \pm x' \pmod{q5^l} \text{(by the Shadowing lemma)}$$

$$\implies (x+x')(x+x') \equiv 0 \pmod{q^2 5^{2l}}$$

or

$$\implies (x-x')(x-x') \equiv 0 \pmod{q^2 5^{2l}}.$$

$$\implies \Sigma(d,l,q) \ll 2\#\{(x,x') : x \neq x',\ x\cdot x' = d \pmod{q^2 5^{2l}}\}$$

recall

$$\sum_{|e|<d,\, e=d \pmod{q^2 5^{2l}}} \{(x,x') : x\cdot x' = e\} \ll_\epsilon d^\epsilon(1+\frac{d}{q^2 5^{2l}}).$$

■

Proof of the equidistribution statement. Recall:

$$\mathrm{dev}_d(\bar{x}) = \frac{|\mathrm{red}_q^{-1}(\bar{x})|}{|\mathscr{H}_d|/|\mathscr{H}_d(q)|} - 1$$

$$\mathrm{dev}_d(\bar{x}) \to 0 \text{ as } d\to\infty,\ d \equiv \pm 1 \pmod 5\ d \text{ fund. disc.}.$$

Sketch of proof. Assume $B_\delta = \{\bar{x} : \mathrm{dev}_d)\bar{x})$
$gt\delta\}$. Lower bound by counting the number of paths that lie in $\mathrm{red}_q^{-1}(B_\delta)$.

Upper bound by the expansion property of $\mathscr{H}_d(q)$. ∎

Lecture 20 29/11/2018

Theorem 3.32 Linnik. *Fix $\nu, \delta > 0$ and let $q \leq d^{\frac{1}{4}-\nu}$, $(q, 30) = 1$. Then the fraction of $\bar{x} \in \mathscr{H}_d(q)$ for which*

$$|\operatorname{dev}_d(\bar{x})| > \delta$$

tends to 0 as $d \longrightarrow \infty$ through $d \equiv \mp 1 \pmod 5$.

Recall 3.3.
Last time

Proposition 3.33 *Let*

$$\Sigma(d, l, q) = \#\{(x, x') \in \mathscr{H}_d^2 : \gamma_x^{(l)} = \gamma_{x'}^{(l)}\}$$

then

$$\Sigma(d, l, q) \ll_\epsilon |\mathscr{H}_d| + d^\epsilon \left(1 + \frac{d}{q^2 5^{2l}}\right).$$

Proof of theorem. Let

$$B_\delta = \{\bar{x} \in \mathscr{H}_d(q) : \operatorname{dev}_d(\bar{x}) > \delta\}$$

and assume that

$$B_\delta \geq \eta |\mathscr{H}_d(q)|$$

for some $\eta > 0$.
Observe that

$$\frac{1}{|\mathscr{H}_d(q)|} \sum_{x \in \mathscr{H}_d} |\gamma_x^{(l)} \cap B_\delta| = (2l+1) \frac{|\operatorname{red}_q^{-1}(B_\delta)|}{|\mathscr{H}_d|}$$

or

$$\frac{1}{|\mathscr{H}_d(q)|} \sum_{x \in \mathscr{H}_d} \frac{|\gamma_x^{(l)} \cap B_\delta|}{(2l+1)} = \frac{|\operatorname{red}_q^{-1}(B_\delta)|}{|\mathscr{H}_d|} \tag{3.1}$$

Choose an l s.t.

$$\frac{1}{5} |\mathscr{H}_d| \leq q^2 5^{2l} \leq 5 |\mathscr{H}_d|$$

note that we can indeed choose such an l.
By definition every $\bar{x} \in B_\delta$ satisfies

$$\operatorname{dev}_d(\bar{x}) > \delta$$

implies

$$\frac{|\operatorname{red}_q^{-1}(\bar{x})|}{|\mathscr{H}_d| / |\mathscr{H}_d(q)|} > 1 + \delta.$$

$$|\operatorname{red}_q^{-1}(B_\delta)| \geq \frac{|B_\delta||\mathscr{H}_d|}{|\mathscr{H}_d(q)|}(1+\delta)$$

so

$$\frac{1}{|\mathscr{H}_d|} \sum_{x \in \mathscr{H}_d} \frac{|\gamma_x^{(l)} \cap B_\delta|}{(2l+1)} > \frac{|B_\delta|}{|\mathscr{H}_d(q)|}(1+\delta)$$

by the assumption that

$$B_\delta \geq \eta |\mathscr{H}_d(q)|$$

we have this is

$$\geq \frac{|B_\delta|}{|\mathscr{H}_d(q)|} + \eta\delta.$$

Let

$$\Phi = \left\{ x \in \mathscr{H}_d : \frac{|\gamma_x^{(l)} \cap B_\delta|}{(2l+1)} > \frac{|B_\delta|}{|\mathscr{H}_d(q)|} + \frac{\eta\delta}{2} \right\}$$

Note that

$$\mathscr{H}_d = \Phi \sqcup \Phi^c$$

$$\frac{1}{|\mathscr{H}_d|} \sum_{x \in \mathscr{H}_d} \frac{|\gamma_x^{(l)} \cap B_\delta|}{(2l+1)} = \frac{1}{|\mathscr{H}_d|} \left(\sum_{x \in \Phi} \underbrace{\frac{|\gamma_x^{(l)} \cap B_\delta|}{(2l+1)}}_{\leq 1} + \sum_{x \in \Phi^c} \underbrace{\frac{|\gamma_x^{(l)} \cap B_\delta|}{(2l+1)}}_{\leq \frac{|B_\delta|}{\mathscr{H}_d(q)|} + \frac{\eta\delta}{2}} \right)$$

$$\leq \frac{1}{|\mathscr{H}_d|} \left(|\Phi| + \overbrace{|\Phi^c|}^{=|\mathscr{H}_d|-|\Phi|} \left(\frac{|B_\delta|}{\mathscr{H}_d(q)|} + \frac{\eta\delta}{2} \right) \right)$$

$$\implies |\Phi| \geq \frac{\eta\delta}{2} |\mathscr{H}_d|. \tag{3.2}$$

Now we count the number of marked non-backtracking paths on $\mathscr{H}_d(q)$ of length $2l+1$ (write $MNBP(\mathscr{H}_d(q))$ for this quantity), which in addition satisfy

$$\frac{|\gamma_x^{(l)} \cap B_\delta|}{(2l+1)} > \frac{|B_\delta|}{|\mathscr{H}_d(q)|} + \frac{\delta\eta}{2}.$$

By (3.2) we have $\eta\delta|\mathscr{H}_d|/2$ of these paths "upstairs".
Recall Siegel implies

$$d^{\frac{1}{2}-\epsilon} < |\mathscr{H}_d| < d^{\frac{1}{2}+\epsilon}$$

so upstairs we have $\gg_\epsilon \frac{\eta\delta}{2} d^{\frac{1}{2}-\epsilon}$ paths.
Claim 1: The previous proposition implies not mant of these paths give the same path mod q

$$MNBP(\mathscr{H}_d(q)) \gg_{\epsilon,\delta\eta} d^{\frac{1}{2}-\epsilon}.$$

On the other hand the total number of marked non-backtracking paths on

$$\mathscr{H}_d(q)$$

is

$$|\mathscr{H}_d(q)| 6 \cdot 5^{2l-1}$$

also observe that $|\mathscr{H}_d(q)| \ll q^2$ so the total number $\sim q^2 5^{2l} \ll d^{1/2+\epsilon}$

$$\frac{1}{5}|\mathscr{H}_d| < q^2 5^{2l} < 5|\mathscr{H}_d|$$

Final input is a Chernoff type bound. ∎

Proposition 3.34 *Fix $\eta, \epsilon > 0$ for any subset*

$$B \subseteq \mathscr{H}_d(q)$$

s.t.

$$|B| \geq \eta |\mathscr{H}_d(q)|$$

the fraction of non-backtracking paths of length $2l+1$ *satisfying*

$$\left| \frac{|\gamma \cap B|}{2l+1} - \frac{|B|}{|\mathscr{H}_d(q)|} \right| \geq \epsilon$$

is

$$\ll_{\epsilon,\eta} e^{-cl}$$

for some $c = c(\epsilon, \eta) > 0$.

Claim 2: This proposition implies that the fraction

$$MNBP(\mathscr{H}_d(q)) \ll d^{\frac{1}{2}-\tau}$$

for some $\tau > 0$. Then

$$\gg_\epsilon d^{\frac{1}{2}-\epsilon}$$

$$\ll_\epsilon d^{\frac{1}{2}-\tau}$$

give a contradiction.

We now check some claims made above.

Claim 3.35 Claim 1. *Bound on* $\#\{(x,x') : \gamma_x^{(l)} = \gamma_{x'}^{(l)}\}$ *gives a bound on the number of distinct paths.*

Proof.

$$\#\{(a,a') \in A^2 : a \equiv a' \pmod{q}\} = \sum n_{a_i}$$

gives a lower bound on $A \pmod{q}$. To show this note that the

$$|A(q)| = |A| - \sum_{i=1}^{|A|} \frac{n_{a_i} - 1}{n_{a_i}} - \sum_{i=1}^{|A|} \frac{1}{n_{a_i}}$$

where $n_{a_i} = \#\{a \in A : a \equiv a_i \pmod{q}\}$. ∎

Claim 3.36 Claim 2. *Follows from two points*

$$d^{\frac{1}{2}-\epsilon} \ll q^2 5^{2l} \ll d^{\frac{1}{2}+\epsilon}$$

$$q < d^{\frac{1}{4}-\nu}$$

3.2 The adelic picture

Lecture 21 4/12/2018

Notation: $\mathbf{A}$ is the ring of adeles of $\mathbf{Q}$

$$\prod_p{}' \mathbf{Q}_p$$

$$\mathbf{A}_f = \prod_{p \text{ finite}}{}' \mathbf{Q}_p$$

so that

$$\mathbf{A} = \mathbf{R} \times \mathbf{A}_f.$$

$$\widetilde{\mathbf{Z}}\colon \varprojlim_n \mathbf{Z}/n = \prod \mathbf{Z}_p.$$

$$\widehat{\mathbf{Z}} \subseteq \mathbf{A}_f$$

maximal compact.

The genus of a quadratic space. For us,

$$L \subseteq \mathbf{Q}^3$$

a lattice, then

$$\mathrm{GL}_3(\mathbf{A}_f) \ni g = \prod g_p$$

$$g \cdot L = \{\alpha \subseteq \mathbf{Q}^3 : \alpha_p = g_p L_p \ \forall p\}$$

$$\mathrm{genus}_{\mathrm{SO}_3}(\mathcal{L}_p) = \mathrm{SO}_3(\mathbf{A}_f)\mathcal{L}_0$$

$$\mathcal{L}_0 = \mathbf{Z}^3,\ a^2 + b^2 + c^2.$$

Proposition 3.37 *There is one* $\mathrm{SO}_3(\mathbf{Q})$ *orbit of* $\mathcal{L}_0$*. i.e.*

$$\mathrm{SO}_3(\mathbf{A}_f) = \mathrm{SO}_3(\mathbf{Q})\,\mathrm{SO}_3(\widehat{\mathbf{Z}})$$

Proof. Based on quaternions having class number 1. ∎

More definitions:

$$P = \{(\mathcal{L}, x) : \mathcal{L} \in \mathrm{genus}_{\mathrm{SO}_3}(\mathcal{L}_0), x \in \mathcal{L},\ x \cdot x = d\}$$

$$P_{(q)} = \{(\mathcal{L}, \bar{x}) : \mathcal{L} \in \mathrm{genus}_{\mathrm{SO}_3}(\mathcal{L}_0), \bar{x} \in \mathcal{L}/q\mathcal{L},\ \bar{x} \cdot \bar{x} = d \pmod q\}$$

$$\widetilde{\mathscr{H}_d} \simeq \mathrm{SO}_3(\mathbf{Q})\backslash P$$

$$K_f[q] = \ker(\mathrm{SO}_3(\widehat{\mathbf{Z}}) \to \mathrm{SO}_3(\mathbf{Z}/q\mathbf{Z}))$$

$$K'_f[q] = \{g \in \mathrm{SO}_3(\widehat{\mathbf{Z}}) : g\bar{x}_q = \bar{x}_q\}$$

we fix a base point $\bar{x}_q \in \mathscr{H}_d(q)$. Then

$$\widetilde{\mathscr{H}_d}(q) \simeq \mathrm{SO}_3(\mathbf{Q})\backslash P_{(q)} \simeq \mathrm{SO}_3(\mathbf{Q})\backslash\, \mathrm{SO}_3(\mathbf{A}_f)/K'_f[q]$$

here we are heavily using the exceptional isomorphism

$$\mathrm{SO}_3 \simeq \mathrm{PGL}_2\,.$$

Claim 3.38

$$\mathrm{SO}_3(\mathbf{Z})\backslash\mathscr{H}_d(q) \simeq \mathrm{SO}_3(\mathbf{Q})\backslash P_{(q)}$$

Proof sketch.

$$\mathrm{SO}_3(\mathbf{Q}) \circlearrowright P_{(q)} \text{ diagonally}$$

$$\mathrm{SO}_3(\mathbf{A}_f) = \mathrm{SO}_3(\mathbf{Q})\,\mathrm{SO}_3(\widehat{\mathbf{Z}})$$

every orbit has a representation $\simeq (\mathcal{L}_0, \bar{x})$, $\bar{x} \in \mathscr{H}_d(q)$.

$$(\mathcal{L}_0, \bar{x}) \simeq (\mathcal{L}_0, \bar{x}')$$

$$\iff$$

$$\exists \gamma \in \underbrace{\mathrm{SO}_3(\widehat{\mathbf{Z}}) \cap \mathrm{SO}_3(\mathbf{Q})}_{\mathrm{SO}_3(\mathbf{Z})}$$

∎

Claim 3.39

$$\mathrm{SO}_3(\mathbf{Q})\backslash P_{(q)} \simeq \mathrm{SO}_3(\mathbf{Z})\backslash \mathscr{H}_d(q)/K'_f[q]$$

Proof sketch. Observe

$$\mathrm{SO}_3(\mathbf{A}_f) \circlearrowright P_{(q)}$$

via

$$\mathcal{L} \otimes \widehat{\mathbf{Z}}/q\mathcal{L} \otimes \widehat{\mathbf{Z}}$$

This action is transitive, i.e.

$$P_{(q)} = \mathrm{SO}_3(\mathbf{A}_f)[\mathcal{L}_0, \bar{x}_q]$$

■

Remark 3.40

$$\mathrm{SO}_3(\mathbf{Z})\backslash S^3 \simeq \mathrm{SO}_3(\mathbf{Q})\backslash \mathrm{SO}_3(\mathbf{A})/K_\infty \mathrm{SO}_3(\widehat{\mathbf{Z}})$$

What about

$$\mathrm{SO}_3(\mathbf{Z})\backslash \mathscr{H}_d?$$

Fix another base point $x_0 \in \mathscr{H}_d$

$$\mathrm{SO}_3(\mathbf{Z})\backslash \mathscr{H}_d \simeq \mathrm{SO}_{x_0}(\mathbf{Q})\backslash \mathrm{SO}_{x_0}(\mathbf{A}_f)/\mathrm{SO}_{x_0}(\widehat{\mathbf{Z}})$$

where SO_{x_0} is the stabiliser of x_0.

$$\begin{array}{c} \mathscr{H}_d \circlearrowright \text{ class gp.} \\ \big\downarrow \\ \mathscr{H}_d(a) \circlearrowright \mathrm{SO}_3(\mathbf{A}_f) \end{array}$$

More concretely

$$\mathrm{SO}_{x_0} : \mathbf{Q}^3 \simeq B^{(0)}(\mathbf{Q})$$
$$\mathrm{SO}_3 \simeq PB^\times$$
$$\mathrm{SO}_{x_0} \simeq PB^\times_{x_0} \simeq \langle a + bx_0 \rangle / Z$$
$$\simeq \mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_m/\mathbf{G}_m$$

where $K = \mathbf{Q}(\sqrt{-d})$.

Recall

$$K^\times \backslash \mathbf{A}_{K,f}^\times / \widehat{O}_K^\times \simeq \mathrm{Pic}(O_K).$$

The graph structure on $\mathscr{H}_d(q)$. $\mathrm{SO}_3(\mathbf{A}_f)$ is a large group, but so is $K'_f[q]$. What is indeed true is

$$\mathscr{H}_d(q) \cong \Gamma_{(3,q)} \mathrm{SO}_3(\mathbf{Q}_5)/K_5$$

analogous to

$$\mathbf{H} = \mathrm{SO}_2 \setminus \mathrm{SL}_2(\mathbf{R})/\mathrm{SL}_2(\mathbf{Z}).$$

Where

$$\Gamma_{(3,q)} PB^\times(\mathbf{Q}) \cap PB^\times(\mathbf{Q}_5) K_f[3,q]$$
$$K_f[3,q] = K_f[3] \cap K'_f[q] \subseteq K'_f[q]$$

Now

$$\mathrm{PGL}_2(\mathbf{Q}_5)/\mathrm{PGL}_2(\mathbf{Z}_5)$$

is a Bruhat-Tits tree, a 6-regular tree.

Automorphic spectrum of PGL_2 of $\mathbf{F}_q(t)$. Analogous to $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$.
Reference I. Efrat: Automorphic spectrum on the tree of PGL_2
Setup: $k = \mathbf{F}_q$, $v_\infty = -\deg$ so $|f|_\infty = q^{\deg(f)}$.

$$K = (k(t))_\infty = k((\frac{1}{t}))$$

$$\mathcal{O}_\infty = \mathcal{O} = k[[t^{-1}]]$$
$$\Gamma = \mathrm{PGL}_2(\mathbf{F}_q[[t]])$$

Analogies:
$$\mathbf{H} \leftrightsquigarrow \mathrm{PGL}_2(K)/\mathrm{PGL}_2(\mathcal{O}) = X$$
$$\mathrm{SL}_2(\mathbf{Z}) \leftrightsquigarrow \Gamma$$

Problem: decompose $L^2(\Gamma\backslash X)$. In the case of

$$\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H} \rightsquigarrow L^2(\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}) = \underbrace{R}_{\text{Residual}} \oplus \underbrace{C}_{\text{cusp forms}} \oplus \underbrace{E}_{\text{Eisenstein}}.$$

Theorem 3.41
$$L^2(\Gamma\backslash X) = R \oplus E$$
$\dim(R) = 2$, $\dim(E) = \infty$ *i.e. no cusp forms.*

Little bit on X (a $(q+1)$-regular tree). Ref: Serre "Trees".
Description

1. In terms of lattices
$$L = \{av_1 + bv_2 : a, b \in \mathcal{O}\}$$
s.t.
$$L \otimes_{\mathcal{O}} K \simeq K^2.$$
Matrix
$$\begin{pmatrix} v_1 & v_2 \end{pmatrix} \mathrm{GL}_2(K)$$
changing basis implies L a class in $\mathrm{GL}_2(K)/\mathrm{GL}_2(\mathcal{O})$.
$$L_1 \sim L_2$$
if $\exists a \in K^\times$ s.t. $L_1 = aL_2$.

$$X = \{\text{equivalence classes of lattices}\}$$
A class L is adjacent to $L' \iff \exists$ representations L_0, L_0' s.t. $L_0' \subseteq L_0$ and $L_0/L_0' \simeq \mathbf{F}_q$.

2. Group theoretic
$$B = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} / K$$
Fact: $\mathrm{PGL}_2(K) = B \cdot \mathrm{PGL}_2(\mathcal{O})$. Each coset $b\,\mathrm{PGL}_2(\mathcal{O})$ has a representative
$$\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$$

n unique and $x \in K$.

Vertices of X

$$\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix} \mathrm{PGL}_2(\mathcal{O})$$

$$\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix} \sim_{\text{adjacent}} \left\{ \begin{pmatrix} t^{n-1} & \xi t^n + x \\ 0 & 1 \end{pmatrix},\ \xi \in \mathbf{F}_q \begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix} \right.$$

Fact:

$$\Gamma\backslash X = \text{chain } \bullet - \bullet - \bullet - \cdots$$

Measures: Normalise Haar on PGL_2 s.t. $\mu(\mathrm{PGL}_2(\mathcal{O})) = q(q-1)$. $\rightsquigarrow$ On $\Gamma\backslash X$, μ gives vertices weights

$$n = 0 : \frac{1}{q+1}$$

$$n = 1 : \frac{1}{q}$$

$$n = 2 : \frac{1}{q^2}$$

$$\vdots$$

$$n : \frac{1}{q^n}$$

Remark 3.42 We have

$$\left\langle f, g \right\rangle_\mu = \int_{\Gamma\backslash X} f \cdot \bar{g} \, \mathrm{d}\mu = \frac{1}{q+1} f(0)\bar{g}(0) + \sum_{i=1}^{\infty} \frac{f(i)\bar{g}(i)}{q^i}.$$

Fact 3.43 *The algebra of operators on functions on X that commute with the automorphisms of X is generated by*

$$(T(f))(s) = \sum_{s' \sim s} f(s')$$

Remark 3.44

$$\Delta = (q+1)\mathrm{id} - T$$

and T descends to $\Gamma\backslash X$.

Given $f : \Gamma\backslash X \simeq \mathbf{N} \to \mathbf{C}$

$$(T(f))(n) = \begin{cases} (q+1)f(1), & n = 0 \\ qf(n-1) + f(n+1), & n > 0 \end{cases}$$

T is self-adjoint with respect to $\langle , \rangle_\mu$.

Eigenfunctions of T.

$$(Tf)(n) = \lambda \cdot f(n),\ \forall n \in \mathbf{N}.$$

$$\implies \lambda f(0) = (q+1)f(1)$$
$$\lambda f(n) = qf(n-1) + f(n+1),\ \forall n > 0$$

i.e.

$$\begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix} = \begin{pmatrix} \lambda & -q \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} f(1) \\ f(0) \end{pmatrix}.$$

Now fix a normalization

$$f(0) = q+1$$
$$f(1) = \lambda.$$

Characteristic values:

$$x_1, x_2 = \frac{-\lambda \pm \sqrt{\lambda^2 - 4q}}{2}$$

note $x_1 = x_2 \iff \lambda = \pm 2\sqrt{q}$.

Eigenfunctions

$$f_\lambda(n) = \begin{cases} \frac{1}{\sqrt{\lambda^2-4q}} \left(\lambda(x_1^n - x_2^n) - q(q+1)(x_1^{n-1} - x_2^{n-1})\right), & n > 0 \\ q+1, & n = 0 \end{cases}$$

for $\lambda \neq \pm 2\sqrt{q}$ note $f_{q+1} = q+1$.

$$f_{2\sqrt{q}}(n) = (q+1-(q-1)n)q^{n/2}$$

$$f_{-2\sqrt{q}}(n) = (-1)^n f_{2\sqrt{q}}$$

We now consider the space $L^2(\Gamma\backslash X)$ with the operator T.

Question: which the f_λ are in $L^2(\Gamma\backslash X)$?

Proposition 3.45 *The only $f_\lambda \in L^2$ with $|\lambda| > 2\sqrt{q}$ are $\lambda = \pm(q+1)$.*

Proof. Normalise f_λ by

$$(x_2 - x_1)f = x_1^{n-1}(\lambda x_1 - q(q+1)) - x_2^{n-1}(\lambda x_2 - q(q+1))$$

assume WLOG $|x_1| > \sqrt{q} \implies |x_2| < \sqrt{q}$ (since $|\lambda| > 2\sqrt{q}$) so $x_2^n = o(q^{n/2})$. So $\lambda x_1 - q(q+1) = 0$ (to be in L^2). So $\lambda = \pm q(q+1)$. ∎

What about the rest?

Note: $|\lambda| < 2\sqrt{q} \implies x_1 = \bar{x}_2$.

Renormalise: Let $x_1 = \sqrt{q}e^{i\theta}, x_2 = \sqrt{q}e^{-i\theta}$.

$$\lambda = 2\sqrt{q}\cos\theta,\ 0 < \theta < \pi$$

$$\tilde{f}_\theta(n) = \frac{x_1 - x_2}{2\sqrt{q}} f_{2\sqrt{q}\cos\theta}(n)$$

so

$$\tilde{f}_\theta(n) = \begin{cases} q^{n/2} i(\sin(n+1))\theta - q\sin(n-1)\theta \\ (q+1)i\sin\theta \end{cases}$$

Proposition 3.46

$$\tilde{f}_\theta \notin L^2$$

for $0 < \theta < \pi$.

Spectral decomposition of $L^2(\Gamma\backslash X)$**.** Continuous spectrum: Extend $\tilde{f}_\theta$ to an odd function $-\pi < \theta < \pi$. Given $\psi \in L^2(\Lambda\backslash X)$ consider

$$F_\psi(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(\theta)\tilde{f}_\theta(n)\,\mathrm{d}\theta.$$

Theorem 3.47 $F_\psi \in L^2(\Gamma\backslash X)$.

$$\left\langle F_\psi, F_\phi \right\rangle_\mu = \left\langle \psi, \phi \right\rangle \frac{1}{2\pi}\int_0^{\pi} \psi(x)\overline{\phi(x)}((q-1)^2 - 4q\sin^2(x))\,\mathrm{d}x$$

Theorem 3.48 *Let* $g \in L^2(\Gamma\backslash X)$*. Then*

$$g(n) = \left\langle g, u_1 \right\rangle u_1(n) + \left\langle g, u_2 \right\rangle u_2(n) + 2\pi\int_0^{\pi} \left\langle g, \tilde{f}_\theta \right\rangle \tilde{f}_\theta \frac{\mathrm{d}\theta}{(q-1)^2 - 4q\sin^2\theta}$$

where

$$u_1 = \sqrt{\frac{q^2-1}{2q}},\ u_2 = \sqrt{\frac{q^2-1}{2q}}(-1)^n.$$