CS137 Seminar 16/1/14 **Alex Best, Peter Davies & Marcin Jurdziński**

1. How many subsets of $\{1, 2, \dots, 2n\}$ are there which contain:

   (i) All of the odd numbers up to $2n$.

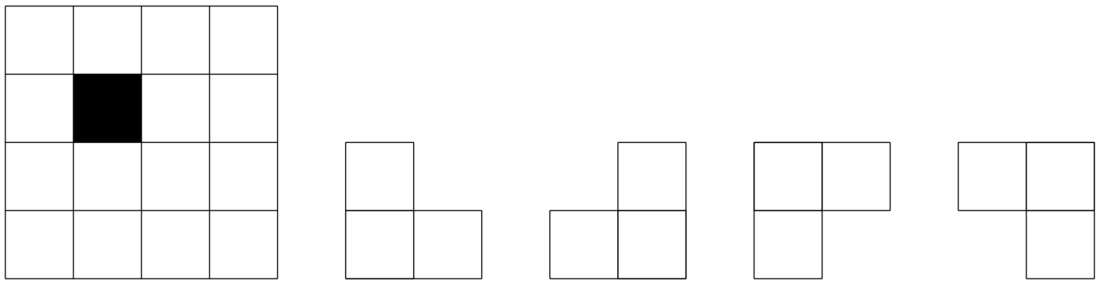
   (ii) At least one odd number.

   (iii) As many odd numbers as even numbers.

   (iv) At least as many odd numbers as even numbers.

   (You don't need a closed expression for the last two).

2. Given a checkerboard of dimension $2^n \times 2^n$ with one square already covered, show that we can cover all the remaining squares with L shaped pieces:



3. Given sets A and B of size m and n respectively find, in all cases, the number of

   (i) Bijections $f\colon A \to B$.

   (ii) Injections $f\colon A \to B$.

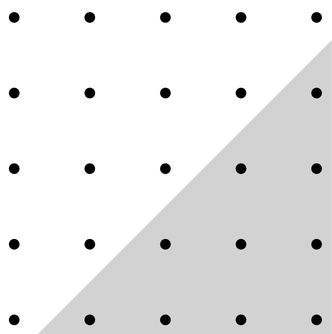
   We have not considered surjections above, and with good reason, the formulae are far harder to determine!

4. What is wrong with the following proof by induction that all cows are the same colour:

   Base case: 1 cow is the same colour as itself.

   Inductive step: given n cows, consider two different groups of $n - 1$ cows. By the inductive assumption, all cows within these groups are the same colour, and since the groups overlap all n cows must be the same colour.

5. (i) Given a grid of $(n + 1) \times (n + 1)$ dots find an expression for the number of dots in the bottom right half (not including the lead diagonal).



(ii) Prove the formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

(iii) (Bonus) Use this technique to find closed forms for the sums

$$\sum_{i=1}^{n} i^2 \text{ and } \sum_{i=1}^{n} i^3.$$

6. Show that given a full binary tree T we have the inequality

$$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1.$$

7. Prove, using the well-ordering principle, that $\sqrt{k}$ is irrational for any prime number k.

8. Given a set P of n people in how many ways can a delegation $D \subset P$ with a specified leader $l \in D$ be chosen from the set.

   For example with two people A and B the answer is 4 as we can either take just A or B as the delegation and they are then the only possible leaders, or we can send them both as the delegation and have either as the leader.

9. In how many ways can n rooks be placed on a chess board of size $n \times n$ such that no two are attacking each other (a rook can only attack pieces in the same row or column as itself).

10. The Cantor-Schroder-Bernstein theorem states that if there is an injection $f : A \to B$ and an injection $g : B \to A$, then there is a bijection $h : A \to B$. Using this fact, prove that the set of all subsets of natural numbers and the set of all infinite subsets of natural numbers have the same cardinality.