1. How many subsets of $\{1, 2, \ldots, 2n\}$ are there which contain:
   (i) All of the odd numbers up to $2n$.
   (ii) At least one odd number.
   (iii) As many odd numbers as even numbers.
   (iv) At least as many odd numbers as even numbers.

   (You don't need a closed expression for the last two).

2. In how many different ways can the following be done:
   (i) A tower built from 10 white bricks and 10 black bricks?
   (ii) A tower built from n white bricks and m black bricks?
   (iii) n rooks be placed on a chess board of size $n \times n$ such that no two are attacking each other (a rook can only attack pieces in the same row or column as itself).

3. Show that given a *full* binary tree T we have the inequality
$$\mathrm{size}(T) \leq 2^{\mathrm{height}(T)+1} - 1.$$

4. Prove the following identities (try to find both a combinatorial and algebraic proof for both of them):
   (i)
$$\binom{n}{2} + \binom{n+1}{2} = n^2.$$
   (ii)
$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

5. Alice has 10 distinct balls. She first splits them into two piles, then chooses a pile with at least two balls in and splits it into two more piles. She repeats this until she has all of the balls in different piles.
   (i) How many steps does it take for Alice to finish doing this.
   (ii) Show that the number of different ways she could do this is
$$\binom{10}{2}\binom{9}{2}\cdots\binom{3}{2}\binom{2}{2}.$$

6. Show that
$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

   (Bonus) Find and prove a similar identity for products of three consecutive numbers.

7. The Cantor-Schröder-Bernstein theorem states that if there is an injection $f\colon A \to B$ and an injection $g\colon B \to A$, then there is a bijection $h\colon A \to B$. Using this fact, prove that the set of all subsets of natural numbers and the set of all infinite subsets of natural numbers have the same cardinality.