The following bounds from lectures will be used in this sheet

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.$$

1. For each of the following pairs of numbers decide which is larger:

   (a)
   $$8,000,000,000 \text{ and } \binom{100}{10}.$$

   (b)
   $$200 \text{ and } \binom{8}{6}.$$

2. You are invited to bet on the result of a lottery. There are 18 differently numbered balls from which you are asked to choose 6 of. The balls are then mixed up and 6 are drawn at random, if you guessed all 6 correctly you win £1,000,000, but get anything wrong and you will win nothing!

   It costs £1 to enter the lottery, should you do so?

3. The bounds written above are good, but not perfect, in fact for fixed n they are the worst when $\binom{n}{k}$ is at a maximum, i.e. for coefficients of the form $\binom{n}{n/2}$. Using the fact that the maximum element of a set of natural numbers is at most as large as the sum of them derive a better upper bound for
   $$\binom{2n}{n}.$$

4. Alice has 10 distinct balls. She first splits them into two piles, then chooses a pile with at least two balls in and splits it into two more piles. She repeats this until she has all of the balls in different piles.

   (i) How many steps does it take for Alice to finish doing this.

   (ii) Show that the number of different ways she could do this is
   $$\binom{10}{2}\binom{9}{2}\cdots\binom{3}{2}\binom{2}{2}.$$

5. Show that
   $$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

   (Bonus) Find and prove a similar identity for products of three consecutive numbers.