1. Solve the recurrence

$$a_0 = 1,$$
$$a_k = 3a_{k-1} + 4^k.$$

2. Find a generating function for the number of ways to make k pence with 1p, 2p 5p 10p, 20p and 50p coins.

3. Let F_n be the Fibonacci sequence ($F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ \forall n \geq 2$).

   (i) Find a simpler form for the generating function $F(x)$ of F_n.

   (ii) Show that

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \cdots .$$

4. Let F_n be the Fibonacci numbers (as above). Show that:

   (i) $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$,

   (ii) F_n divides F_{2n} for all n,

   (iii) (Bonus) F_n divides F_{kn} for all n and k.

5. Given a positive integer n let $p(n)$ be the number ways (ignoring orderings) of splitting n up as a sum of positive numbers. These are called *partitions* of n.

   For example $p(4) = 5$ as it can be written as $4,\ 3+1,\ 2+2,\ 2+1+1$ and $1+1+1+1$.

   (i) Find a simpler form for the generating function of $p(n)$.

   (ii) Find a simpler form for the generating function of $p_o(n)$, the number of partitions of n as a sum of only odd numbers.

   (iii) Prove that $p_o(n) = p_d(n)$, the number of partitions of n into sums of distinct terms (each summand is used at most once).

   (iv) (Bonus) Show that the number of partitions into numbers not divisible by 3 is the same as the number of partitions where each summand is used no more than twice.