BUNTES Raynaud and Runge

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Raynaud's theory of formal R-schemes is one branch of expanding rigid analytic geometry. The other is Berkovich's theory. Conrad puts it as Raynaud's theory focuses on more algebraic aspects whereas Berkovich's theory allows more point-wise intuition and cohomological techniques. For example, Berkovich's spaces actually carry a classical topology instead of a G-topology. Today, we focus on Raynaud's perspective.

We've developed rigid analytic spaces, its coherent modules, and their cohomology. One difficulty of the current set-up is that there is no natural way to associate a rigid space to some kind of reduction to its residue field. As one would expect, having access to this kind of reduction reveals more algebraic aspects of our space and might even allow us to translate geometric facts regarding a rigid space over a local field to some corresponding space over a residue field (Abhyankar). The tool that allows this kind of reduction is Raynaud's formal schemes.

1 Formal Schemes

We start with R a complete valuation ring with quotient field K and residue field k. We denote by R_n the space $R/(\pi^n)$ where $\pi \in R$ is some element such that $0 < |\pi| < 1$ (if R is a dvr, then π can be taken as a uniformizer).

The main idea of Raynaud's framework is to shift our focus from K-algebras to R-algebras. One can think of this as shifting our focus from $\mathbb{Q}_p$ to $\mathbb{Z}_p$. But to make this theory more rich, we want to have rings that are complete with respect to our norms. The analogy would be taking the completion of $\mathbb{Z}$ with respect to the p-adic norm to get $\mathbb{Z}_p$.

Our rigid analytic spaces locally look like an affinoid space, i.e. the Max spec of an affinoid algebra of the form

$$K\{X_1, \dots, X_n\}/I$$

where $K\{X_1, \dots, X_n\} = \{\sum a_i X^i \mid |a_i| \to 0 \text{ as } \|i\| \to \infty\}$. As one would expect, working over the field K doesn't naturally allow us to create a reduction mod π where π is a uniformizer. However, we can try to create a corresponding modle over R which does allow such a reduction.

Definition 1.1. A topologically finitely presented (tfp) R-algebra is of the form

$$R\{X_1, \dots, X_n\}/I$$

where we have

$$R\{X_1, \dots, X_n\} = \{\sum a_i X^i \mid |a_i| \to 0 \text{ as } \|i\| \to \infty\}$$

and I is a finitely generated ideal.

A tfp R-algebra is called admissible if it is R-flat.

Remark 1.2. In our given scenario, an R-module is flat if and only if it is torsion-free.

Notice that we have

$$K \otimes_R R\{X_1, \dots, X_n\} = R\{X_1, \dots, X_n\}[1/\pi] \cong T_n(K).$$

In fact, we have in general that if $\mathscr{A}$ is a tfp R-algebra, then $A = K \otimes_R \mathscr{A}$ is an affinoid algebra.

Remark 1.3. One thing to notice about these R-algebras is that if R is a discrete valuation ring, then $R\{X_1, \dots, X_n\}$ is in fact the inverse limit

$$\varprojlim_n R_n[X_1, \dots, X_n].$$

It is important to note that if R is not a discrete valuation ring, then taking $\mathfrak{m}$ to be the maximal ideal gives $\mathfrak{m}^2 = \mathfrak{m}$, so the inverse limit

$$\varprojlim_n R/\mathfrak{m}^n[X_1, \dots, X_n]$$

actually produces $k[X_1, \dots, X_n]$. This is why the general construction that follows uses a choice of π.

Proposition 1.4. *Let $\mathscr{A}$ be a tfp R-algebra. Then we have the following for any $\pi \in R$ such that $0 < |\pi| < 1$:*

(a) $\mathscr{A}$ is π-adically complete.

(b) $\mathscr{A}[\pi^\infty]$, the π-power torsion of $\mathscr{A}$, is finitely generated as an ideal.

(c) If I is an ideal of $T_n(K)$, then the intersection $I \cap R\{X_1, \dots, X_n\}$ is finitely generated.

(d) Since R-modules are flat iff they are torsion-free, every K-affinoid algebra A is of the form $K \otimes_R \mathscr{A}$ for some admissible R-algebra $\mathscr{A}$.

Notice that part (c) of the above proposition allows us to create an admissible R-algebra from any tfp R-algebra by quotienting out the π-power torsion ideal.

This is a suitable algebra to consider as a model over R when considering an affinoid space over K. However, just taking the Max Spec will be insufficient since we will be getting points over k with little relation to K. Instead, we take the space

$$\operatorname{Spf} \mathscr{A} := \varinjlim_n \operatorname{Spec} \mathscr{A}_n$$

where $\mathscr{A}_n = \mathscr{A}/(\pi^n)$. The corresponding structure sheaf is the projective limit

$$\mathcal{O}_{\mathrm{Spf}\,\mathscr{A}} = \varprojlim_n \mathcal{O}_{\mathscr{A}_n}.$$

Before we proceed, it is good to notice that the projective system of rings that gives rise to the above direct limit are the maps

$$\mathscr{A}/\mathfrak{m}\mathscr{A} \to \mathscr{A}/(\pi^n).$$

Notice that even if R is not a discrete valuation ring, the spectrum of the rings are in bijection. In fact, the underlying set of our topological space is really an affine space over k. The distinction of our construction from this affine space arises from the structure sheaf.

To see the contrast in the structure sheaf, we can observe a principle open set $D(f)$ for some $f \in \mathscr{A}$. Let Let $D(f)$ be the non-vanishing locus of f. Then we get that the ring of section $\Gamma(\mathcal{O}_{\mathrm{Spf}\,\mathscr{A}}, D(f))$ is precisely $\mathscr{A}_{\{f\}}$, the π-adic completion of the localization A_f.

Remark 1.5. A consequence of this identification of the actual topological space is that even if R is not Noetherian (e.g. the ring of integers of $\mathbb{C}_p$), the resulting formal R-scheme is a Noetherian space.

We can then extend this approach to global pictures to get formal schemes.

Definition 1.6. A formal R-scheme X_∞ is a locally topologically ringed space which has a finite open cover by tfp affine formal R-schemes.

Example 1.7. Let $A = R\{x,y\}/(\pi, xy - \pi)$ and $X = \mathrm{Spf}\,A$. As a topological space, X is $\mathrm{Spec}\,k[x,y]/(xy-\pi)$. The structure sheaf, however, contains richer data since it contains elements such as $\sum_{i=0}^{\infty} \pi^i x^i$.

Definition 1.8. The special fiber X_0 of an affine formal R-scheme $X = \mathrm{Spf}\,\mathscr{A}$ is the k-scheme $X_0 = \mathrm{Spec}\,\mathscr{A}/\mathfrak{m}\mathscr{A}$.

Our construction above shows that the special fiber X_0 is topologically identical to the formal R-scheme X. As one might expect, this means that some topological aspects of our theory depends solely on the special fiber.

For example, a formal R-scheme X is separated if each X_n is separated. This is the case if and only if X_0 is separated.

On the other hand, the generic fiber of an R-scheme recovers the corresponding rigid space. Specifically, for affine formal schemes $X = \mathrm{Spf}\,\mathscr{A}$, taking the generic fiber X_η gives the K-affinoid space $\mathrm{Sp}(\mathscr{A} \otimes_R K)$.

This highlights the strength of formal R-schemes: the scheme carries the information of its corresponding rigid analytic space as its generic fiber while also containing a reduction of the rigid analytic space as its special fiber. This relation is more explicitly given as Raynaud's generic fiber functor.

Theorem 1.9. *The assignment* $\mathrm{Spf}\,\mathscr{A}$ *to* $\mathrm{Sp}(K \otimes_R \mathscr{A})$ *from the category of tfp formal affine schemes to affinoid rigid analytic spaces over K is functorial and carries Zariski-open immersions to quasi-compact admissible opens.*

This functor extends uniquely while preserving fiber products and mapping open immersions to admissible opens so that we get a functor between tfp formal R-schemes and rigid spaces over K.

Proposition 1.10. *Raynaud's generic fiber functor maps admissible formal blowups of admissible formal R-schemes to isomorphisms of rigid analytic spaces.*

The idea here is that blow-ups do not affect the generic fiber.

1.1 Projective Line

Start with the affine line $X = \mathrm{Spec}\,R[x]$ over R. We have $X_K = \mathrm{Spec}\,K[x]$ and the analytification is $(X_k)^{an} = (\mathbb{A}^1_K)^{an}$.

The π-adic completion is $\widehat{X} = \mathrm{Spf}\,R\{x\}$ and its generic fiber $\widehat{X}_\eta$ is the closed unit disc $\mathrm{Sp}\,K\{x\}$. The canonical open immersion $\widehat{X}_\eta \to (X_K)^{an}$ is an isomorphism onto the affinoid domain in $(X_K)^{an}$ consisting of points z such that $|x(z)| \le 1$.

Removing the origin O from X gives a scheme X' where $X'_K = X_K$. The formal completion of X', however, is $\widehat{X'} = \mathrm{Spf}\,R\{x,T\}/(xT-1)$. Its generic fiber is then the complement $]O[$ in $\widehat{X}_\eta$.

For the projective line, $\mathbb{P}^1_R = \mathrm{Proj}\,R[x,y]$, the analytic projective line $(\mathbb{P}^1_K)^{an}$ can be realized in different ways. First, consider the usual affine cover of $\mathbb{P}^1_K$ by the charts $U_1 = \mathrm{Spec}\,K[x/y]$ and $U_2 = \mathrm{Spec}\,K[y/x]$. Their analytifications $(U_1)^{an}$ and $(U_2)^{an}$ are infinite unions of closed discs (see Example 6) centered at 0, resp. ∞. (The idea here is that $(\mathbb{A}^1_K)^{an}$ is the direct limit of the system $D \to D \to D \to \cdots$ where the maps are multiplication by some $a \in K$ where $|a| < 1$. The image of each map is $D(0,|a|)$ which produces a rigid K space.) Gluing along the admissible opens $(U_1)^{an} - \{0\}$ and $(U_2)^{an} - \{\infty\}$ in the obvious way, we obtain $(\mathbb{P}^1_K)^{an}$.

On the other hand, we can look at the formal completion $\widehat{\mathbb{P}^1_R}$. Is generic fiber is canonically isomorphic to $(\mathbb{P}^1_K)^{an}$ by Raynaud's generic fiber functor. The separated topologically of finite type formal R-scheme $\mathbb{P}^1_R$

is covered by the affine charts $V_1 = \operatorname{Spf} R\{x/y\}$ and $V_2 = \operatorname{Spf} R\{y/x\}$ whose intersection is given by $V_0 = \operatorname{Spf} R\{x/y, y/x\}/((x/y)(y/x) - 1)$. The generic fibers of V_1 and V_2 are closed unit discs around $x/y = 0$, resp. $y/x = 0$, and $(V_0)_\eta$ coincides with their boundaries. So in this way, $(\mathbb{P}^1_K)^{an}$ is realized as the Riemann sphere obtained by gluing two closed unit discs along their boundaries.

2 References

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