Geometric approaches to solving Diophantine equations

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Tomorrows Mathematicians Today 2013

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Introduction

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  (Pythagorean triples)
- \(x^2 - ny^2 = 1\)  
  (Pell’s equation)
- \(x^3 + 48 = y^4\)
- \(9^x - 8^y = 1\)
What do we want to know?

- Are there any solutions?
- How many are there?
- Can we classify them?
- How can we compute them?
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In this talk

Two general ideas:
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- Geometry of numbers
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Two general ideas:
- Geometry of numbers
- Rational points on surfaces
Minkowski and the Geometry of Numbers

1910 - Hermann Minkowski publishes his paper “Geometrie der Zahlen” and sparks a new field called the Geometry of Numbers.

Hermann Minkowski
A motivating example

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We can try a few primes, and notice that $2^2 + 1^2 = 5$, $3^2 + 2^2 = 13$, $4^2 + 1^2 = 17$, ... all work.
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So maybe $x^2 + y^2 = p$ when $p \equiv 1 \pmod{4}$.

**Theorem (Fermat’s Christmas Theorem)**

An odd prime $p$ can be written as $p = x^2 + y^2$ if and only if $p \equiv 1 \pmod{4}$. 
What can geometry do for us?

When $p = 5$ we can look at points satisfying our criteria:
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Minkowski’s first theorem

We say a set $B$ in $\mathbb{R}^n$ is symmetric if $x \in B \implies -x \in B$. We say a set $B$ is $\mathbb{R}^n$ is convex if $x, y \in B \implies x + \lambda(y - x) \in B$ for $0 \leq \lambda \leq 1$. 

Theorem (Minkowski)

If $B$ is a convex symmetric body and $\Lambda$ a lattice in $\mathbb{R}^n$ then $B$ contains a non-zero point of the lattice if:

$$\text{Vol}(B) > 2^d \text{area of a single cell of the lattice } \Lambda.$$ 

We can find it using determinants, or the order of our lattice as a subgroup of $\mathbb{Z}^n$ for the more algebraically inclined.
Minkowski’s first theorem

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**Theorem (Minkowski)**

*If $B$ is a convex symmetric body and $\Lambda$ a lattice in $\mathbb{R}^n$ then $B$ contains a non-zero point of the lattice if:*

$$\text{Vol}(B) > 2^d i$$

*Where $i$ is the area of a single cell of the lattice $\Lambda$. We can find it using determinants, or the order of our lattice as a subgroup of $\mathbb{Z}^n$ for the more algebraically inclined.*
Let’s apply it

A circle in \( \mathbb{R}^2 \) is symmetric and convex, great!
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A circle in $\mathbb{R}^2$ is symmetric and convex, great!
The area of our lattice cell is $p$.
The area of our circle is $\pi 2p$.
So Minkowski tells us that as:

$$\pi 2p = \text{Vol}(B) > 2^d i = 2^2 p = 4p$$

We have a point which satisfies $x^2 + y^2 = kp$ for some $k \geq 1$, and also $x^2 + y^2 < 2p$. So $x^2 + y^2 = p$ and we are done.
Rational points on surfaces

Euler looked at

$$A^4 + B^4 = C^4 + D^4$$

with $A, B, C, D \in \mathbb{Q}$

This defines a surface in $\mathbb{R}^4$. 
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One parametrisation of solutions[1]:

$$a(s, t) = s^7 + s^5 t^2 - 2s^3 t^4 + 3s^2 t^5 + st^6$$
$$b(s, t) = s^6 t - 3s^5 t^2 - 2s^4 t^3 + s^2 t^5 + t^7$$
$$c(s, t) = s^7 + s^5 t^2 - 2s^3 t^4 - 3s^2 t^5 + st^6$$
$$d(s, t) = s^6 t + 3s^5 t^2 - 2s^4 t^3 + s^2 t^5 + t^7$$

Unfortunately this does not give us all solutions.
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\[ A^4 + B^4 = C^4 + D^4 \]

with \( A, B, C, D \in \mathbb{Q} \).

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One parametrisation of solutions[1]:

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\begin{align*}
    a(s, t) & = s^7 + s^5 t^2 - 2s^3 t^4 + 3s^2 t^5 + st^6 \\
    b(s, t) & = s^6 t - 3s^5 t^2 - 2s^4 t^3 + s^2 t^5 + t^7 \\
    c(s, t) & = s^7 + s^5 t^2 - 2s^3 t^4 - 3s^2 t^5 + st^6 \\
    d(s, t) & = s^6 t + 3s^5 t^2 - 2s^4 t^3 + s^2 t^5 + t^7
\end{align*}
\]

Unfortunately this does not give us all solutions.
Counting solutions

Q: How many solutions are there?
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A: $\infty$
Can we find a better way of counting them?
We want to estimate the density of our solutions.
Heights

We want a different way of describing the size of a rational number. Take

\[ x = \frac{a}{b} \in \mathbb{Q}, \quad a, b \in \mathbb{Z} \]

\[ \gcd(a, b) = 1 \]
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Hurrah! There are only finitely many points with height less than a given value.
Counting points

We now want to count points in a set $X$ with bounded height. We do this in a simple way and define:

$$N(X, B) := \# \{ x \in X | H(x) \leq B \}$$

We can analyse the growth of this function as $B$ grows.
Counting points

We now want to count points in a set $X$ with bounded height. We do this in a simple way and define:

$$N(X, B) := \#\{x \in X | H(x) \leq B\}$$

We can analyse the growth of this function as $B$ grows. Manin and others have conjectured that the growth of $N(X, B)$ is asymptotically governed by geometric properties of the surface for many problems[2].
Morals

Geometry can help us show solutions exist to Diophantine equations. Geometric properties govern the density of the solutions to some problems.
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Geometry can help us show solutions exist to Diophantine equations.
Geometric properties govern the density of the solutions to some problems.
And there is a lot more interplay between these two areas.
References
